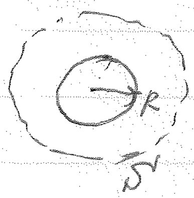
Solving Problems via Gauss Law

Examples

1) spherical shell of charge with uniform surface charge density σ, radius R

solution via Gauss Law Integral form



Let S be spherical surface of radius r centered concentric with shell.
By spherical symmetry of the problem we know that $\vec{E}(\vec{r})$ must be of the form $E(r)\hat{r}$. Everywhere on surface S, the outward normal is $\hat{n} = \hat{r}$.

$$\oint_S d\vec{a} \cdot \vec{E} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta \, r^2 \, \hat{r} \cdot \vec{E}(\vec{r})$$

$$= \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta \, r^2 E(r)$$

since $E(r)$ is constant on surface

$$= 4\pi r^2 E(r) = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow E(r) = \frac{Q_{encl}}{4\pi r^2 \varepsilon_0}$$

Clearly if $r < R$, $Q_{encl} = 0$
if $r > R$, $Q_{encl} = 4\pi R^2 \sigma$

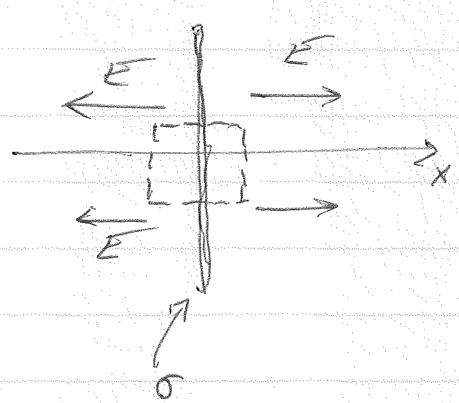
$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < R \\ \dfrac{(4\pi R^2 \sigma)\hat{r}}{4\pi \varepsilon_0 r^2} & r > R \end{cases}$$

like pt charge

Same solution as before!

3) infinite charged plane with uniform surface charge σ

solution via Gauss Law, integral form

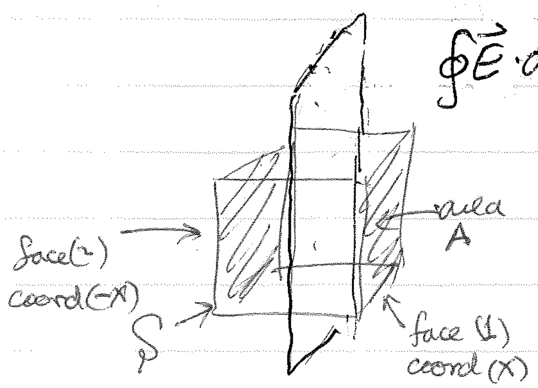


by symmetry we expect that $\vec{E}(\vec{r})$ has the form

$$\vec{E}(\vec{r}) = E(x)\,\hat{x}$$

where $E(x) = -E(-x)$

let S' be the surface of a cube located symmetrically about the plane.



$\oint \vec{E}\cdot d\vec{a}$ only contributions are from the two planes with $\hat{n} = +\hat{x}$ and $\hat{n} = -\hat{x}$. all other surfaces vanish as $\hat{n}\cdot\vec{E} = 0$ on these surfaces.

$$\oint d\vec{a}\cdot\vec{E} = \int_{(1)} da\,\hat{x}\cdot\vec{E} + \int_{(2)} da\,(-\hat{x})\cdot\vec{E}$$

since $\vec{E}(\vec{r}) = E(x)\,\hat{x}$,

$$= \int_{(1)} da\,E(x) - \int_{(2)} da\,E(-x)$$

$$= A\left[E(x) - E(-x)\right] = 2A\,E(x)$$

$$\oint d\vec{a}\cdot\vec{E} = 2A\,E(x) = \frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \quad E(x) = \frac{\sigma}{2\varepsilon_0} \qquad x > 0$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\,\hat{x} & x > 0 \\[2ex] -\dfrac{\sigma}{2\varepsilon_0}\,\hat{x} & x < 0 \end{cases}$$

Note: flip in E as cross plane is

$$\Delta\vec{E} = \left(\frac{\sigma}{2\varepsilon_0}\hat{x}\right) - \left(\frac{-\sigma}{2\varepsilon_0}\hat{x}\right) = \frac{\sigma}{\varepsilon_0}\hat{x} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

Same as for shell

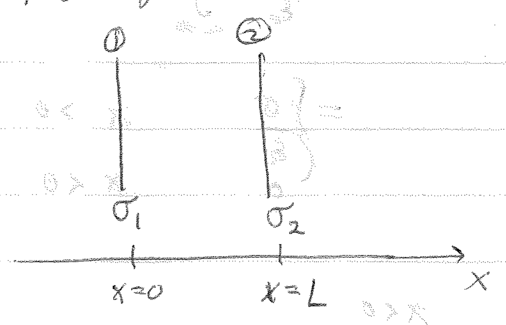
~~We could also solve all these problems by Gauss's Law in differential form.~~

~~(3) charged plane: By symmetry expect~~ $\cancel{\vec{E} \text{ is } E(x)\hat{x}}$

$\cancel{\nabla\cdot\vec{E} = \frac{\partial E}{\partial x} = 0}$ ~~everywhere except on plane~~

$\cancel{\Rightarrow \vec{E} = E_0 \text{ constant}}$

Two planes



use superposition:

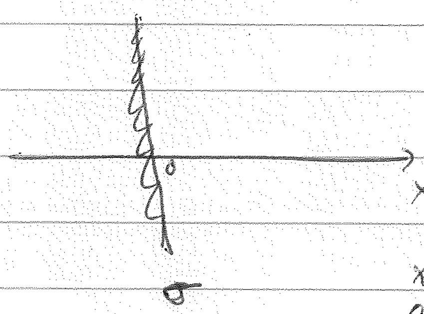
field from ① is $E_1 = \begin{cases} \dfrac{\sigma_1}{2\varepsilon_0}\hat{x} & x > 0 \\[2ex] -\dfrac{\sigma_1}{2\varepsilon_0}\hat{x} & x < 0 \end{cases}$

field from ② is $E_2 = \begin{cases} \dfrac{\sigma_2}{2\varepsilon_0}\hat{x} & x > L \\[2ex] -\dfrac{\sigma_2}{2\varepsilon_0}\hat{x} & x < L \end{cases}$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \begin{cases} \dfrac{\sigma_1 + \sigma_2}{2\varepsilon_0}\hat{x} & x > L \\[2ex] \dfrac{-\sigma_1 - \sigma_2}{2\varepsilon_0}\hat{x} & x < 0 \\[2ex] \dfrac{\sigma_1 - \sigma_2}{2\varepsilon_0}\hat{x} & 0 < x < L \end{cases}$$

if $\sigma_2 = -\sigma_1$ then $\vec{E} = 0$ except between plates – this is a capacitor.

Gauss Law in Differential form



$$\vec{E}(\vec{r}) = E(x)\,\hat{x} \qquad \rho(\vec{r}) = \sigma\,\delta(x)$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \;\Rightarrow\; \frac{\partial E}{\partial x} = \frac{\sigma}{\varepsilon_0}\,\delta(x)$$

$$\int_{-\infty}^{x} dx'\,\frac{\partial E}{\partial x'} = \int_{-\infty}^{x} \frac{\sigma}{\varepsilon_0}\,\delta(x')$$

$$E(x) - E(-\infty) = \frac{\sigma}{\varepsilon_0}\times\begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

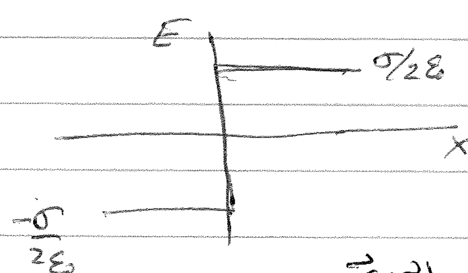
$$E(x) = \begin{cases} E(-\infty) & x<0 \\ E(-\infty)+\frac{\sigma}{\varepsilon_0} & x>0 \end{cases} \qquad \begin{matrix} E \text{ constant for } x<0 \\ \text{" \quad " \quad " } x>0 \end{matrix}$$

If require by symmetry that $E(\infty) = -E(-\infty)$

then have $E(\infty) = E(-\infty) + \frac{\sigma}{\varepsilon_0} = -E(+\infty) + \frac{\sigma}{\varepsilon_0}$

$$2E(\infty) = \frac{\sigma}{\varepsilon_0}$$

$$E(\infty) = \frac{\sigma}{2\varepsilon_0} = -E(-\infty)$$

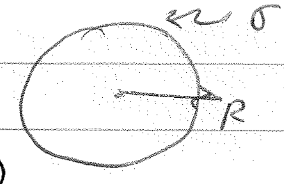


flip in E at σ is $\frac{\sigma}{\varepsilon_0}$

$$\vec{E}(\vec{r}) = \begin{cases} \frac{\sigma}{2\varepsilon_0}\,\hat{x} & x>0 \\ \frac{-\sigma}{2\varepsilon_0}\,\hat{x} & x<0 \end{cases}$$

same solution as before

for a shell of uniform σ at radius R



$$\vec{E}(\vec{r}) = E(r)\hat{r} \qquad \rho(\vec{r}) = \sigma\delta(r-R)$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Longrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E) = \frac{\sigma}{\varepsilon_0}\delta(r-R)$$

$$\frac{\partial}{\partial r}(r^2 E) = \frac{\sigma}{\varepsilon_0} r^2 \delta(r-R)$$

$$\int_0^r dr' \frac{d}{dr'}(r'^2 E) = \frac{\sigma}{\varepsilon_0}\int_0^r r'^2 \delta(r'-R)$$

$$r^2 E(r) - 0\,E(0) = \frac{\sigma}{\varepsilon_0}\begin{cases} 0 & r<R \\ R^2 & r>R \end{cases}$$

$$E(r) = \begin{cases} 0 & r<R \\ \dfrac{\sigma R^2}{\varepsilon_0 r^2} & r>R \end{cases}$$

since $\sigma = \dfrac{Q}{4\pi R^2}$ $\qquad E(r) = \begin{cases} 0 & r<R \\ \dfrac{Q}{4\pi\varepsilon_0 r^2} & r>R \end{cases}$

same solution as before!

for a sphere of uniform ρ_0 up to radius R

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = \begin{cases} \rho_0/\varepsilon_0 & r<R \\ 0 & r>R \end{cases}$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2 E\right) = \rho/\varepsilon_0$$

$$\int_0^r dr'\frac{d}{dr'}\left(r'^2 E\right) = \int_0^r dr'\, r'^2 \frac{\rho(r')}{\varepsilon_0}$$

$$r^2 E(r) - 0\,E(0) = \begin{cases} \dfrac{r^3}{3}\dfrac{\rho_0}{\varepsilon_0} & r<R \\[2ex] \dfrac{R^3}{3}\dfrac{\rho_0}{\varepsilon_0} & r>R \end{cases}$$

$$E(r) = \begin{cases} \dfrac{\rho_0}{3\varepsilon_0} r & r<R \\[2ex] \dfrac{\rho_0 R^3}{3\varepsilon_0}\dfrac{1}{r^2} & r>R \end{cases}$$

using $\rho_0 = \dfrac{Q}{\frac{4}{3}\pi R^3}$ gives $E(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{R^3} & r<R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r^2} & r>R \end{cases}$

Electrostatic potential

Maxwell's Eqn for electrostatics:

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\times\vec{E} = 0$$

Fundamental theorem of vector calculus:

If $\vec{\nabla}\times\vec{E} = 0$ for a vector field $\vec{E}$, then there always exists some scalar function V, such that

$\vec{E} = -\vec{\nabla}V$ $\quad$ V is called the scalar potential for $\vec{E}$

Note V is not unique: if add constant

$$V' = V + V_0$$

then $\vec{\nabla}V' = \vec{\nabla}V + \vec{\nabla}V_0 = \vec{\nabla}V + 0 = -\vec{E}$

Both V and V' are equally good scalar potentials for $\vec{E}$

In electrostatics, V is called the electrostatic potential

Proof: For electrostatics we know this must be so.

Since

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\vec{r}')\,\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

and we had that

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}$$

or $$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

we have

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\epsilon_0}\int d^3r'\,\rho(\vec{r}\,')\;\vec{\nabla}\left(\frac{1}{|r-r'|}\right)$$

$$= -\vec{\nabla}V(\vec{r})$$

where $$\boxed{V(\vec{r}) \equiv \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

Coulomb's law for electrostatic potential.

Easier to compute V than $\vec{E}$, since integral is a scalar, not a vector, integrand.

<u>More general proof</u>:

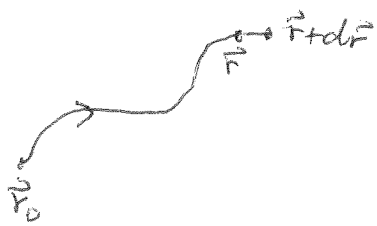
Since $\vec{\nabla}\times\vec{E} = 0$, we have the line integral

$\int_{\vec{r}_0}^{\vec{r}} d\vec{l}\cdot\vec{E}$ is indep of path from $\vec{r}_0$ to $\vec{r}$. (since $\vec{\nabla}\times\vec{E} = 0$ $\Rightarrow \oint d\vec{l}\cdot\vec{E} = 0$ for any closed path)

$$\Rightarrow \text{ define } V(\vec{r}) \equiv -\int_{\vec{r}_0}^{\vec{r}} d\vec{l}\cdot\vec{E}$$ show that $\vec{E} = -\vec{\nabla}V$

<u>by definition of gradient</u>:

$$d\vec{r}\cdot\vec{\nabla}V(\vec{r}) = V(\vec{r}+d\vec{r}) - V(\vec{r})$$



$$= -\int_{\vec{r}_0}^{\vec{r}+d\vec{r}} d\vec{l}\cdot\vec{E} + \int_{\vec{r}_0}^{\vec{r}} d\vec{l}\cdot\vec{E} = -\int_{\vec{r}}^{\vec{r}+d\vec{r}} d\vec{l}\cdot\vec{E}$$

$$= -d\vec{r}\cdot\vec{E}$$

true for any $d\vec{r}$ $\Rightarrow$ $\vec{\nabla}V = -\vec{E}$

In above, $\vec{r}_0$ is arbitrary reference point.

Changing $\vec{r}_0$ to a different $\vec{r}_0{}'$ just adds a constant to V.

$$V'(\vec{r}) = -\int_{\vec{r}_0{}'}^{\vec{r}} d\vec{l}\cdot\vec{E} = \underbrace{-\int_{\vec{r}_0{}'}^{\vec{r}_0} d\vec{l}\cdot\vec{E}}_{\parallel} - \int_{\vec{r}_0}^{\vec{r}} d\vec{l}\cdot\vec{E}$$

$$= V_0 + V(\vec{r})$$ where V_0 is const indep of $\vec{r}$.

Differential equation for V

Since $\vec{E} = -\vec{\nabla}V$, one of Maxwell's eqn's:

$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\vec{\nabla}V = 0$ is automatically satisfied since $\vec{\nabla}\times\vec{\nabla}V = 0$ for any scalar function V

Second Maxwell eqn gives

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\vec{\nabla}V = -\nabla^2 V = \rho/\varepsilon_0$$

$$\boxed{-\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}}$$ Poisson's equation for potential V