Alternative way to write

$$W = \frac{1}{2}\sum_i q_i \sum_{j\neq i} \frac{q_j}{4\pi\varepsilon_0 |\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\sum_i q_i V(\vec{r}_i)$$

potential due to all charges excluding q_i

For a continuous charge distribution $\rho(\vec{r})$, we have

$$\boxed{W = \frac{1}{2}\int d^3r\, \rho(\vec{r})\, V(\vec{r}) = \frac{1}{2}\int d^3r\, d^3r' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|}}$$

where V is potential from ρ

Another way:

$$W = \frac{1}{2}\int d^3r\, \rho(\vec{r}) V(\vec{r}) = \frac{\varepsilon_0}{2}\int d^3r\, (\vec{\nabla}\cdot\vec{E})V$$

using $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

use $\vec{\nabla}\cdot(V\vec{E}) = V\vec{\nabla}\cdot\vec{E} + \vec{E}\cdot\vec{\nabla}V$

$$W = \frac{\varepsilon_0}{2}\int d^3r \left\{ \vec{\nabla}\cdot(V\vec{E}) - \vec{E}\cdot\vec{\nabla}V \right\}$$

use Gauss's Theorem

$$W = \frac{\varepsilon_0}{2}\oint_S d\vec{a}\cdot\vec{E}\,V - \frac{\varepsilon_0}{2}\int d^3r\, \vec{E}\cdot\vec{\nabla}V$$

where S is surface that bounds all the ~~charge~~ system
ie S is surface that goes out to ∞.

As $S \to \infty$

$$\oint_S d\vec{a} \cdot \vec{E} V \to 0.$$ To see this note, for localized charge distribution ρ,

$$\vec{E} \sim \frac{1}{r^2}, \quad V \sim \frac{1}{r}$$

$$\oint_S d\vec{a} \cdot \vec{E} V \sim \int d\theta \int d\phi \sin\theta \; r^2 \left(\frac{1}{r^2}\right)\left(\frac{1}{r}\right)$$

$$\sim \int d\theta \int d\phi \sin\theta \; \frac{1}{r} \to 0$$ as $r \to \infty$ on surface S

So

$$W = \frac{-\varepsilon_0}{2} \int d^3r \; \vec{E} \cdot \vec{\nabla} V$$ use $\vec{E} = -\vec{\nabla} V$

$$= \frac{\varepsilon_0}{2} \int d^3r \; \vec{E} \cdot \vec{E}$$

$$\boxed{W = \frac{\varepsilon_0}{2} \int d^3r \; E^2}$$

Summary

Continuous charge distribution

$$W = \frac{1}{2} \int d^3r \; \rho(\vec{r}) V(\vec{r})$$

$$= \frac{1}{2} \int d^3r \int d^3r' \; \frac{\rho(\vec{r}) \rho(\vec{r}')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|}$$

$$= \frac{\varepsilon_0}{2} \int d^3r \; E^2(\vec{r})$$

point charges

$$W = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\varepsilon_0 |\vec{r}_i - \vec{r}_j|}$$

Self Energy

1) $W = \frac{\varepsilon_0}{2}\int d^3r\ \vec{E}(\vec{r})\cdot\vec{E}(\vec{r})$

2) $\quad = \frac{1}{2}\int d^3r\ \rho(\vec{r})\,V(\vec{r})$

3) $\quad = \frac{1}{2}\int d^3r \int d^3r'\ \frac{\rho(\vec{r})\,\rho(\vec{r}\,')}{4\pi\varepsilon_0\,|\vec{r}-\vec{r}\,'|}$

$\Big\}$ continuum charge distr.

4) $\quad = \frac{1}{2}\sum_{i\neq j} \frac{q_i q_j}{4\pi\varepsilon_0\,|\vec{r}-\vec{r}\,'|}$ for point charges

Question (1) is clearly always ≥ 0
(4) can be ≥ 0 or < 0 depending on charges q_i
But (1) was derived from (4). Where is inconsistency?

Consider several charge distributions $\rho_i(\vec{r})$. What is energy?

Let $V_i(\vec{r})$ = potential from ρ_i ie $\nabla^2 V_i = -\rho_i/\varepsilon_0$
$\vec{E}_i(\vec{r})$ = electric field from ρ_i ie $\vec{\nabla}\cdot\vec{E}_i = \rho_i/\varepsilon_0$

Superposition $\Rightarrow$ total $\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) + \cdots$
total $V(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) + \cdots$

$$\Rightarrow \quad W = \frac{\varepsilon_0}{2}\int d^3r\,(\vec{E}_1+\vec{E}_2+\cdots)\cdot(\vec{E}_1+\vec{E}_2+\cdots)$$

$$= \frac{\varepsilon_0}{2}\int d^3r\left(E_1^2+E_2^2+\vec{E}_1\cdot\vec{E}_2+\vec{E}_2\cdot\vec{E}_1+\cdots\right)$$

$$= \frac{\varepsilon_0}{2}\int d^3r\left(\sum_i E_i^2 + \sum_{i\neq j}\vec{E}_i\cdot\vec{E}_j\right)$$

"self energy" of ρ_i i.e. energy to form distribution ρ_i

interaction energy between ρ_i and ρ_j i.e. energy to move ρ_j into position wrt ρ_i

equivalently:

$$W = \frac{1}{2}\int d^3r\int d^3r'\,\frac{[\rho_1+\rho_2+\cdots][\rho_1+\rho_2+\cdots]}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{2}\int d^3r\int d^3r'\left[\sum_i \frac{\rho_i(\vec{r})\rho_i(\vec{r}')}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|} + \sum_{i\neq j}\frac{\rho_i(\vec{r})\rho_j(\vec{r}')}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}\right]$$

self energy of ρ_i

interaction energy between ρ_i and ρ_j

Suppose we apply above for a set of point charges

$\rho_i(\vec{r}) = q_i\,\delta(\vec{r}-\vec{r}_i)$ ← pt charge q_i at position $\vec{r}_i$

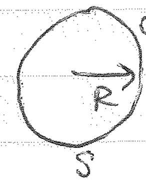
$$W = \frac{1}{2}\sum_i \frac{q_i^2}{4\pi\varepsilon_0|\vec{r}_i-\vec{r}_i|} + \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{4\pi\varepsilon_0|\vec{r}_i-\vec{r}_j|}$$

self energy $\to\infty$

this is (4)

So inconsistancy resolved by noting that (4) did not include the infinite self energy of pt charge.

Another way to see that self energy of pt charge $\to \infty$.
Model pt charge as a spherical shell of radius R,
with total charge q distributed uniformly on surface.
What is the electrostatic energy of this configuration?



$\sigma = \frac{q}{4\pi R^2}$

Method ①

$$W = \frac{1}{2}\int d^3r\, \rho(\vec{r})\, V(\vec{r})$$

$$= \frac{1}{2}\int_S da\, \sigma\, V(R)$$

↑ potential on surface $V(R) = \frac{q}{4\pi\varepsilon_0 R}$

$$= \frac{1}{2}\underbrace{4\pi R^2}_{\text{area sphere}}\sigma\,\frac{q}{4\pi\varepsilon_0 R}$$

$$W = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 R} = \frac{q^2}{8\pi\varepsilon_0 R}$$

Method ② $\quad W = \frac{\varepsilon_0}{2}\int d^3r\, \vec{E}(\vec{r})\cdot\vec{E}(\vec{r})$

$$\vec{E}(\vec{r}) = \begin{cases} \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} & r>R \\ 0 & r<R \end{cases}$$

$$\Rightarrow W = \frac{\varepsilon_0}{2}\int_R^\infty dr\int_0^\pi d\theta\int_0^{2\pi} d\phi\, r^2\sin\theta\, \frac{q^2}{(4\pi\varepsilon_0 r^2)^2}$$

$$= \frac{4\pi\varepsilon_0}{2}\frac{q^2}{(4\pi\varepsilon_0)^2}\int_R^\infty dr\,\frac{1}{r^2}$$

$$\boxed{W = \frac{q^2}{8\pi\varepsilon_0 R}}$$ same as by ①

as $R \to 0$, and sphere shrinks down to a pt particle,
$W \to \infty$ as $1/R$

Is self energy of pt charge something to worry about?

No: If pt charges are never created or destroyed, then this infinite self energy will always cancell out on both sides of any energy balance equation.

yes: From relativity we know that an energy behaves like a mass via $W = mc^2$ - does infinite self energy at a point charge electron mean it would act like it has infinite mass?

This leads to second question - where is energy located?

① $W = \frac{1}{2}\int d^3r \, \rho(r) V(r)$ ← looks like energy is located at charges because contrib to W is zero where $\rho = 0$

② $W = \frac{\varepsilon_0}{2}\int d^3r \, \vec{E}(\vec{r})\cdot\vec{E}(\vec{r})$ ← looks like energy is located in electric field $\vec{E}$ because there is contrib to W wherever $E \neq 0$.

for our spherical shell model of pt charge, ① says energy lies on shell ② says energy is located throughout all space outside shell.

Although both ① and ② give same result for W, it is important to decide "where's the energy located" to have a complete theory of gravitation.
From general relativity we know that

all mass is a source of gravitational attraction, and all energy W acts like a mass W/c^2. ① locates this mass only at location of pt particle ② ~~but~~ locates this mass distributed throughout space - need to know where energy is located to say how sources of gravitational attraction are distributed.

In electrostatics, no way to decide where energy is located. In electrodynamics we will find that ② is correct interpretation - energy located in $\vec{E}$.

Return now to problem of self energy. If energy is located in electric field, then when we try to move the pt charge, we have also to move the electric field surrounding it. The effective mass of moving this electric field is $\frac{\epsilon_0}{2c^2}\int d^3r\, E^2$, which would diverge in limit $R \to 0$ of true pt charge.

This gives chance to understand the origin of mass! Maybe all the electron's mass is really the mass contained in the electric field that surrounds the electron! Perhaps electron is not a true point charge, but has radius R just right so that $\frac{\epsilon_0}{2c^2}\int d^3r\, E^2$ = observed rest mass of electron.

How big is this R?

We found $W = \frac{\varepsilon_0}{2}\int d^3r\, E^2 = \frac{q^2}{8\pi\varepsilon_0 R}$

Set $q = e$ electron charge $\rightarrow W = m_e c^2$

$$\Rightarrow m_e = \frac{e^2}{8\pi\varepsilon_0 R c^2} \quad \cancel{\Rightarrow R = \frac{8\pi\varepsilon_0 m_e c^2}{e^2}}$$

$$\Rightarrow R = \frac{e^2}{8\pi\varepsilon_0 m_e c^2}$$

Define $a_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} = 2.82\times 10^{-13}$ cm "classical electron radius"

[factor $\frac{1}{2}$ difference between R and a_e ~~comp~~ comes from specific model we used - the spherical ~~shell~~ shell. If we used a different model - ie. a uniformly charged sphere - we would get a different factor $O(1)$. In any case, a_e sets the magnitude of the length scale]

10^{-13} cm is typical size of atomic nucleus.

Above arguement suggests, that if classical electrostatics (ie Coulomb's Law) holds correct down to the smallest distances, the electron charge should be smeared out over a distance of order a_e.

Alternatively, if electron is a true point particle, Coulomb's law should fail once one get down to length scales $\sim a_e \sim 10^{-13}$ cm.

However, all experimental evidence to date indicates that Coulomb's law ___does___ hold down to the smallest length scales, and electron really is a pt charge. This has been tested down to length scale $\sim 10^{-17}$ cm from electron-positron scattering.

~~There really is no self consistant theory of electromagnetism which~~
~~This leads to a fundamental inconsistancy in~~

This problem of the self energy of a pt charge leads to a fundamental inconsistancy of electrodynamics, that doesn't get fixed even if one includes quantum mechanics. Tricks have been devised (Renormalization theory) to deal with this self energy in a way that one can calculate all observable quantities to very high precision. But conceptual problem remains.

Trick amounts to

$$m_{expt} = \left(m_{field} + m_{mech} \right)$$

$\frac{W}{c^2}$ from electric field energy

"mechanical mass"

since $m_{field} > m_{expt}$ as radius electron $< 10^{-13}$ cm, we need to choose $m_{mech} < 0$.

But can never measure m_{field} or m_{mech} separately, only the sum m_{expt} can be measured