Computing electrostatic pressure by Virtual Work

If we have a system in initial state with total energy U_i then do something to it so that it winds up with total energy U_f, then the work-energy theorem of mechanics says that

$$U_f - U_i = W \text{ is the work done on the system.}$$

Consider a spherical conductor of radius R with uniform σ on surface. The initial electrostatic energy is

$$U_i = \frac{1}{2}\int d^3r\, \rho V = \frac{1}{2}\int d^3r\, \sigma\,\delta(r-R)\, V = \frac{1}{2}4\pi R^2 \sigma\, V(R)$$

since $V(\vec{r}) = V(r)$ is spherically symmetric and depends only on $r = |\vec{r}|$. Since $4\pi R^2 \sigma = Q$ and for $r \geq R$ $V(r) = \frac{Q}{4\pi\varepsilon_0 r}$ we have $U_i = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R}$

Now imagine expanding the sphere to radius $R+dR$ keeping Q constant. Now the electrostatic energy is

$$U_f = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0(R+dR)} = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R}\frac{1}{\left(1+\frac{dR}{R}\right)} \simeq \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R}\left(1-\frac{dR}{R}\right)$$

so $$\Delta U = U_f - U_i = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R}\left(1-\frac{dR}{R}\right) - \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R}$$

$$= -\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R^2}dR = W$$

energy decreases $\Rightarrow$ negative work done on system (ie system does work on its environment)

Now, from thermodynamics, if we have a system and change its volume then the work done on the system must be $-p\,dV = W$.

So $W = -p\,dV = -\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R^2}dR$

change in volume of sphere is

$$dV = dR\frac{d}{dR}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2 dR \Rightarrow dR = \frac{dV}{4\pi R^2}$$

So

$$W = -p\,dV = -\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R^2}\frac{dV}{4\pi R^2}$$

$$p\,dV = \frac{1}{2\varepsilon_0}\left(\frac{Q}{4\pi R^2}\right)^2 dV = \frac{1}{2\varepsilon_0}\sigma^2 dV$$

$$\Rightarrow \boxed{p = \frac{1}{2\varepsilon_0}\sigma^2}$$

just as we expect from our general result

pressure pushes surface of sphere outwards which is why electrostatic energy decreases when R increases.

(2.5.4) Capacitance

If have two conductors, one with $+Q$ net charge and the other with $-Q$, and $\Delta V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E}\cdot d\vec{l}$ is the potential difference between the (+) conductor and the (−) conductor, then define the capacitance

$$\boxed{C = \frac{Q}{\Delta V}}$$

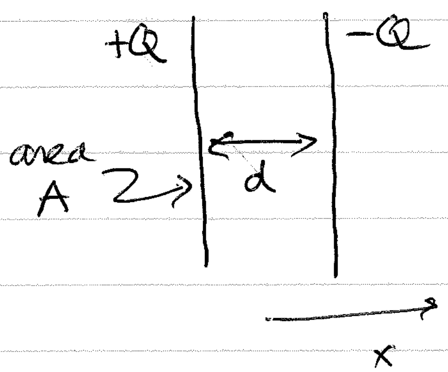
units: E has units $\frac{\text{nt}}{\text{coul}}$

V has units $\frac{\text{nt-m}}{\text{coul}} = \text{volt}$

C has units $\frac{\text{coul}}{\text{volt}} = \text{farad}$

Turns out that since $V \sim Q$, C is independent of Q and depends only on the geometry of the conductors

Example Parallel Plate Capacitor



electric field is zero outside

$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{x}$ in between where $\sigma = \frac{Q}{A}$

potential difference

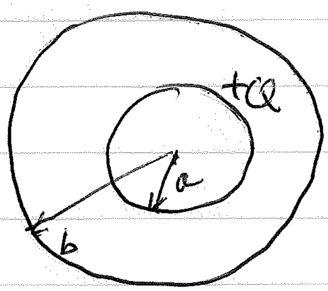
$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{l} = \int_+^- \vec{E}\cdot d\vec{l}$$

$$= \int_0^d dx\, \hat{x}\cdot\vec{E} = \frac{\sigma}{\varepsilon_0} d = \frac{dQ}{\varepsilon_0 A}$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{\left(\frac{dQ}{\varepsilon_0 A}\right)} = \boxed{\frac{\varepsilon_0 A}{d} = C}$$

Does not matter what path of integration is as long as starts on conductor with $+Q$ and ends on conductor with $-Q$. This is because conductors are at constant potential and because $\nabla\times\vec{E} = 0$ then $\oint \vec{E}\cdot d\vec{l}$ depends only on endpoints and not the path

Example Capacitance of two concentric spherical shells



$$\Delta V = V(r=a) - V(r=b)$$

$$= -\int_b^a d\vec{l}\cdot\vec{E} = \int_a^b d\vec{l}\cdot\vec{E}$$

choose outward radial path $\quad = \int_a^b dr\ \hat{r}\cdot\vec{E}$

But $\vec{E}(\vec{r}) = \begin{cases} 0 & r<a \\ \dfrac{Q\hat{r}}{4\pi\varepsilon_0 r^2} & a<r<b \\ 0 & b<r \end{cases}$

$$\Delta V = \int_a^b dr\ \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)_a^b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon_0\, ab}{b-a}}$$

Energy stored in a capacitor

Consider building up the charge on the capacitor by adding dq to the (+) conductor by taking it from the (−) conductor

work done to transfer dq is $\quad dW = (V_+ - V_-)dq$

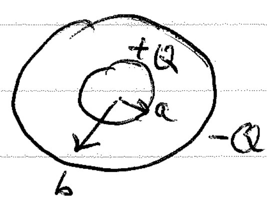
$$= \Delta V \, dq$$

Now $\quad C = \dfrac{Q}{\Delta V} \quad$ so $\quad \Delta V = \dfrac{Q}{C}$

$$W = \int dW = \int_0^Q dq \, \frac{q}{C} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C \Delta V^2$$

$$\boxed{W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C \Delta V^2}$$

Lets compute the energy stored in a spherical capacitor by several different ways



(i) $\quad W = \dfrac{\varepsilon_0}{2} \int d^3r \, E^2$

$$E = \begin{cases} \dfrac{Q \, \hat{r}}{4\pi\varepsilon_0 r^2} & a < r < b \\ 0 & \text{otherwise} \end{cases}$$

$$W = \frac{\varepsilon_0}{2} 4\pi \int_a^b dr \, r^2 \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{\varepsilon_0}{2} \frac{4\pi \, Q^2}{(4\pi\varepsilon_0)^2} \int_a^b dr \, \frac{1}{r^2}$$

$$W = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)_a^b = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{(b-a)}{ab}$$

(2) $$W = \frac{1}{2}\int d^3r\, \rho V = \frac{1}{2}\left[+Q\,V(a) - Q\,V(b)\right]$$

$$= +\frac{1}{2}Q\left[V(a) - V(b)\right] = \frac{1}{2}Q\,\Delta V = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

using ΔV that we found when computed C

(3) $$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{(b-a)}{ab}$$ using $C = \frac{4\pi\varepsilon_0\, ab}{b-a}$

All methods give the same answer!

Special Methods For Solutions

$$V(\vec{r}) = \int \frac{d^3r}{4\pi\varepsilon_0} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

In principle, this is solution to all electrostatics

In practice, often hard to evaluate integral, even if know $\rho(\vec{r})$. But often even harder, because we often don't ~~know~~ know what is $\rho(\vec{r})$ – for example, when there is conductor in problem, we may not know (unless we solve for explicitly) what the surface charge induced on the conductor is – but we do know that surface of conductor is equipotential $V = \text{const}$.

$\Rightarrow$ Often it is easier to directly solve for V via

Poisson's Eqn:

$$-\nabla^2 V = \rho/\varepsilon_0$$

with "boundary conditions" on V at the surface of conductors. Can then use this solution, together with $\frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0}$ to determine the induced charge density on conductor's surface.

Often want to solve problem in which $\rho = 0$ everywhere, except on conductor surfaces.

Laplace's Eqn

$$\Rightarrow \quad \nabla^2 V = 0$$

with $V = \text{const}$ on conductor surface

Solutions to Laplace's eqn are called harmonic functions

Harmonic functions have the properties

1) $V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V(\vec{r}\,')\, da'$

$\vec{r}\,'$ on surface of sphere of radius R centered at $\vec{r}$

$\Rightarrow V(\vec{r})$ is average value of V over surface of sphere centered at $\vec{r}$

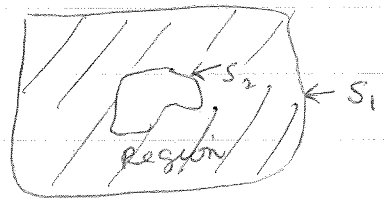
Proof: see text

2) (1) $\Rightarrow$ V can have no local max or min. extrema of V occur at boundaries

Uniqueness theorems

1) Solution to Laplace's eqn in some region of space is uniquely determined, if the value of V is specified on all boundaries of region

ie. ex:



~~E.g. $\nabla^2 V=0$ in shaded region and $V(\vec{r}) = V_1(\vec{r})$ on surface S_1 $= V_2(\vec{r})$ on surface S_2~~

Want $\nabla^2 V = 0$ in ~~Often~~ shaded region. Want V to have given values on surfaces S_1 and S_2, ie $V(\vec{r}) = g_1(\vec{r})$ for $\vec{r}$ on S_1, $V(\vec{r}) = g_2(\vec{r})$ for $\vec{r}$ on S_2

If find solution $V(\vec{r})$ that has these properties and some one else finds $V'(\vec{r})$ that also has these properties then $V(\vec{r}) = V'(\vec{r})$ ie solution is unique

proof: If there exist such V and V', consider
$V'' \equiv V - V'$. Then $V''=0$ on boundaries
and $\nabla^2 V''=0$. Since max and min of
V'' must be on boundaries $\Rightarrow V''=0$
everywhere!

2) In region containing conductors, and filled with given ρ, electric field $\vec{E}$ is uniquely determined if total charge on each conductor is given.

Proof: see text.

Uniqueness theorems are important because they tell us that if we find a particular solution to the type of electrostatic problem described in the theorem, then it is the only solution.

Method of image charges:

At surface of conductor, we know $\vec{E}$ must be normal. This will be true if the surface of the conductor is a surface of mirror reflection symmetry.

So if have charges q_i on one side of conducting plane, put "imaginary" charges in symmetrical positions