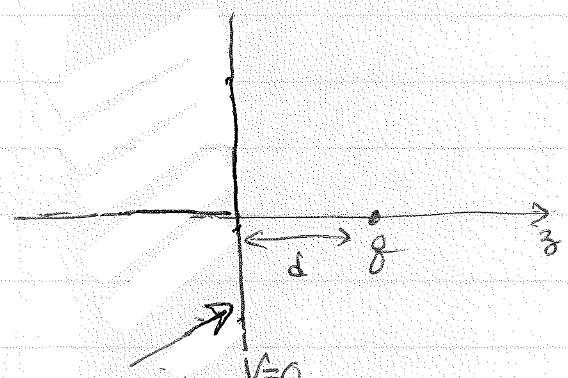
Image charge Method



infinite conducting plane, grounded to V=0

boundary value problem

① $\nabla^2 V = -\frac{\rho}{\varepsilon_0} = -\frac{q}{\varepsilon_0}\,\delta^3(\vec{r}-d\hat{z})$

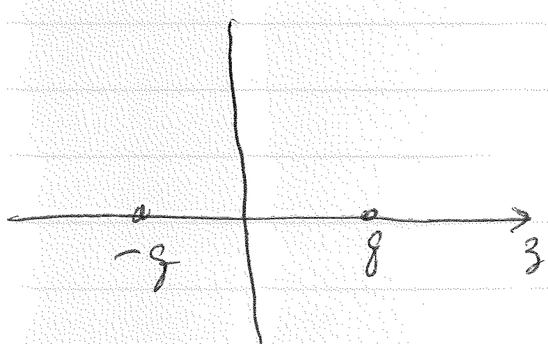
on right of plane

② $V=0$ on the plane

(ie on the boundary of the region to right of plane)

If we find a solution, it is the unique solution.

Solution: put fictitious "image charge" $-q$ at d to left of plane



$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-d\hat{z}|} - \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{r}+d\hat{z}|}$$

$$= \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right\}$$

$$V(x,y,0) = 0$$

also $\nabla^2 V(\vec{r}) = -\frac{q}{\varepsilon_0}\,\delta^3(\vec{r}-d\hat{z})$

$+\frac{q}{\varepsilon_0}\,\delta^3(\vec{r}+d\hat{z})$

but $\delta^3(\vec{r}+d\hat{z}) = 0$ everywhere on right side plane

$\Rightarrow \nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ on right side

So V is the solution

Since we have conducting plane, we can find electric field to right of plane, and use it to find surface charge induced on plane.

$$\bar{E} = -\vec{\nabla} V$$

$$\vec{E}(x,y,0) = \frac{\sigma(x,y)}{\varepsilon_0}\hat{n} = \frac{\sigma(x,y)}{\varepsilon_0}\hat{z}$$

$$\Rightarrow \sigma(x,y) = \varepsilon_0 E_z(x,y,0)$$

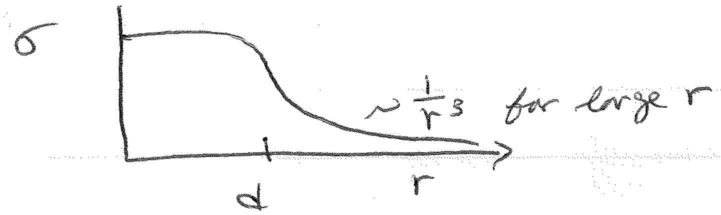
$$= -\varepsilon_0 \frac{\partial V}{\partial z}(x,y,0)$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{q}{4\pi\varepsilon_0}\left\{\left(\frac{-1}{2}\right)\frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \left(\frac{-1}{2}\right)\frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

$$E_z(x,y,0) = +\frac{q}{4\pi\varepsilon_0}\frac{-d-d}{[x^2+y^2+d^2]^{3/2}} = \frac{-qd}{2\pi\varepsilon_0[x^2+y^2+d^2]^{3/2}}$$

$$\sigma(x,y) = -\frac{qd}{2\pi}\frac{1}{[x^2+y^2+d^2]^{3/2}} = -\frac{qd}{2\pi}\frac{1}{[r^2+d^2]^{3/2}}$$

where $\sqrt{x^2+y^2} = r$ is radial dist in plane of conductor



Total charge induced on the grounded plane is

$$q_{ind} = \int_{plane} da\, \sigma = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \sigma(x,y) = \int_0^{2\pi} d\phi \int_0^{\infty} dr\, r\, \sigma(r,\phi)$$

$$= 2\pi \int_0^{\infty} dr \left(\frac{-qd}{2\pi}\right) \frac{r}{(r^2+d^2)^{3/2}} = -qd \left[\frac{-1}{(r^2+d^2)^{1/2}}\right]_0^{\infty}$$

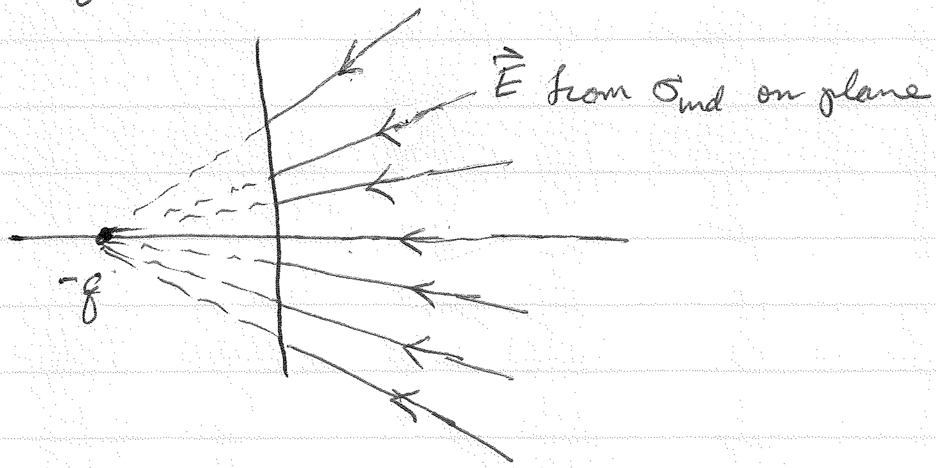
$$= -qd\left[0 - \frac{-1}{d}\right] = -q$$

That the plane is grounded at $V=0$ means that it is attached to a battery (charge reservoir) that can transfer or remove charge from the plane, so as to always keep its potential $V=0$, with respect to pt $r \to \infty$. In present problem, battery transfers charge $-q$ to the conducting plane.

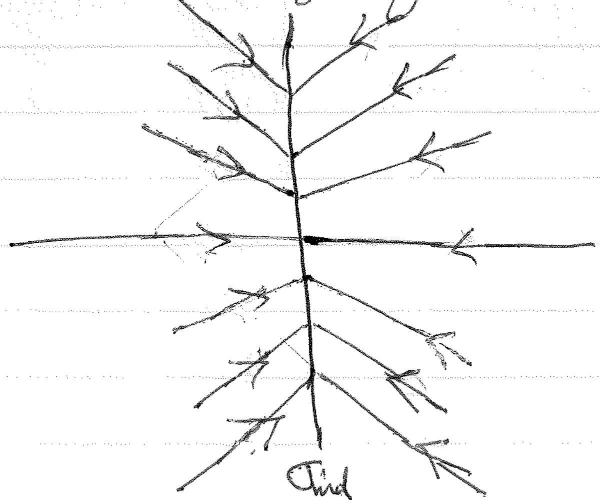
Image charge method solves for V only in region of interest, contained within the specified boundary. It does not give V on the outside of the boundary.

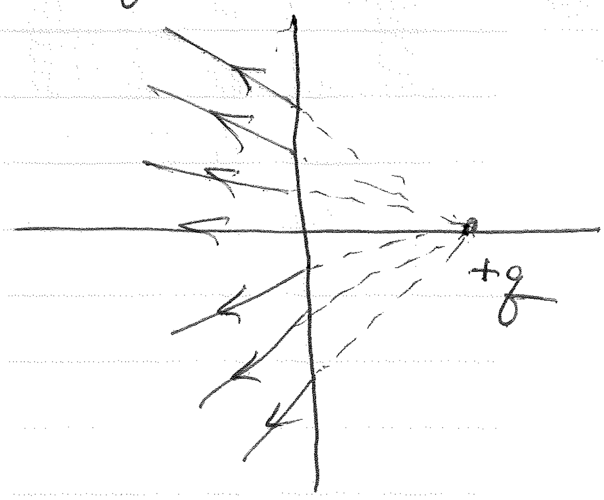
In the present example, we can find $\vec{E}$ to left of plane, by using symmetry arguments

$\vec{E}$ to right of plane, due to σ_{ind} on plane, is equal to field of image charge.



By symmetry, the $\vec{E}$ $\overbrace{\text{on left of plane}}^{\text{due to } \sigma_{ind}}$ is just mirror image of $\vec{E}$ on right of plane due to σ_{ind}.



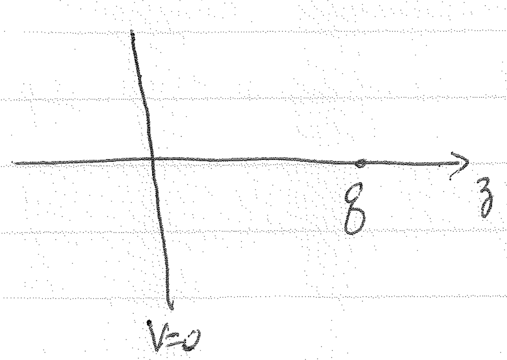


$\vec{E}$ here looks just like field from pt charge $-q$ at $+d\hat{z}$

add to $\vec{E}$ from real charge $+q$ at $d\hat{z}$

$\Rightarrow$ total $\vec{E}$ on left of plane vanishes.

Force on charge in front of grounded conductor:



Force on q is force due to all other charges in system. In this case it is due to the σ_{ind} on the conducting plane

$\vec{F} = q\vec{E}_{ind}$ but $\vec{E}_{ind}$ is same as field produced by a $-q$ at $-d\hat{z}$

$$= q\left[\frac{-q}{4\pi\epsilon_0}\frac{(d\hat{z} - (-d\hat{z}))}{|d\hat{z} - (-d\hat{z})|^3}\right]$$

where $d\hat{z}$ is location of q and $-d\hat{z}$ is location of image

$$= \frac{-q^2\, 2d\,\hat{z}}{4\pi\epsilon_0 (2d)^3}$$

$$\vec{F} = \frac{-q^2}{16\pi\epsilon_0 d^2}\hat{z}$$ attracts q to the plane.

Work done to move q in from infinity to the pt $\vec{r} = d\hat{z}$

$$W = -\int_\infty^d d\vec{l}\cdot\vec{F} = -\int_\infty^d dz\, F_z$$ choose path along z axis

$$= \int_d^\infty dz\left(\frac{-q^2}{16\pi\epsilon_0}\right)\frac{1}{z^2} = \frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{z}\right)_d^\infty$$

$$\boxed{W = \frac{-q^2}{16\pi\varepsilon_0 d}}$$

$W < 0$ means energy is released as charge moves to plane – this is because electrostatic force is attractive

Note, W above is <u>not</u> the energy that would be if the image charge was really present. The energy of a $+q$ and $-q$ seperated by distance $2d$

is $W = \dfrac{-q^2}{4\pi\varepsilon_0 (2d)} = \dfrac{-q^2}{8\pi\varepsilon_0 d}$ twice that above.

~~We can explain the difference as follows:~~

~~$W = \frac{\varepsilon_0}{2}$~~