Linear Dielectrics

For some materials there may be a frozen in $\vec{P} \neq 0$ even when $\vec{E} = 0$ (we call these ferroelectrics in analogy to ferromagnets)

But in most materials $\vec{P} = 0$ when $\vec{E} = 0$ and when $\vec{E}$ is small, $\vec{P}$ is linear in $\vec{E}$

$$\vec{P}(\vec{r}) = \varepsilon_0 \chi_e \vec{E}(\vec{r})$$

"electric susceptibility"
comes from atomic and molecular polarizability

Such a material is called a <u>linear dielectric</u>

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}$$

we call $K \equiv 1 + \chi_e$ the "dielectric constant"

$\varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon_0 K$ the "permitivity" of the dielectric

In a vacuum, $\chi_e = 0$ and $\varepsilon = \varepsilon_0$ the permitivity of free space

$$\boxed{\vec{D} = \varepsilon \vec{E}}$$

$$K = 1 + \chi_e = \varepsilon / \varepsilon_0$$

For a linear dielectric

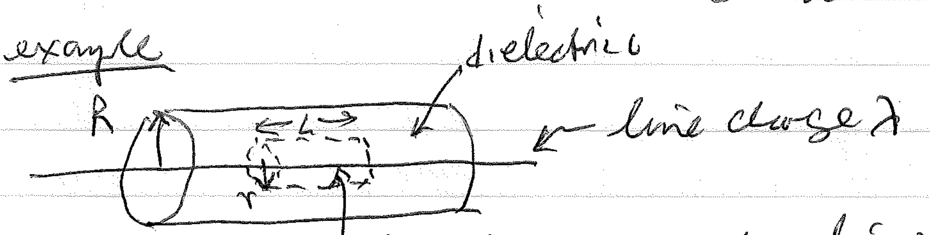
$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(\varepsilon_0 \chi_e \vec{E}) = -\vec{\nabla}\cdot\left(\varepsilon_0 \chi_e \frac{\vec{D}}{\varepsilon}\right)$$

$$= -\frac{\varepsilon_0}{\varepsilon}\chi_e \vec{\nabla}\cdot\vec{D} = -\frac{\chi_e}{K}\rho_f = \frac{-\chi_e}{1+\chi_e}\rho_f$$

$$\boxed{\rho_b = \frac{-\chi_e}{1+\chi_e}\rho_f}$$

when there is no free charge density ρ_f, then there is no bound charge density ρ_b.

example



line charge λ surrounded by cylindrical dielectric of radius R, permitivity ε.

We expect the fields to have cylindrical symmetry, and point outwards in the cylindrical radial direction

$\Rightarrow$ induced $\vec{P}$ is radial outwards $\Rightarrow \vec{D}$ is radial outwards

so $\vec{D}(\vec{r}) = D(r)\hat{r}$ $\quad$ $\hat{r}$ is cylindrical radial direction

$$\oint_S d\vec{a}\cdot\vec{D} = 2\pi r L\, D(r) = Q^{free}_{enclosed} = \lambda L$$

$$D(r) = \frac{\lambda}{2\pi r} \Rightarrow \boxed{\vec{D}(\vec{r}) = \frac{\lambda}{2\pi r}\hat{r}}$$

works for all r inside or outside dielectric

$$\Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon} = \begin{cases} \dfrac{\lambda\hat{r}}{2\pi\varepsilon r} & r<R \text{ inside dielectric} \\[2ex] \dfrac{\lambda\hat{r}}{2\pi\varepsilon_0 r} & r>R \text{ outside dielectric} \end{cases}$$

We see that $\vec{D}(\vec{r})$ is continuous at the surface of the dielectric but $\vec{E}(\vec{r})$ is not!

$$\underset{\uparrow \atop \text{just outside}}{\vec{E}(R^+)} - \underset{\uparrow \atop \text{just inside}}{\vec{E}(R^-)} = \frac{\lambda}{2\pi R}\left[\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right]\hat{r} = \frac{\lambda}{2\pi\varepsilon_0 R}\left[1 - \frac{1}{K}\right]\hat{r}$$

If there is a jump in $\vec{E}$ it must be because there is a surface charge on the surface of the dielectric

$$\varepsilon_0 \hat{r}\cdot\left[\vec{E}(R^+) - \vec{E}(R^-)\right] = \frac{\lambda}{2\pi R}\left[1 - \frac{1}{K}\right] = \sigma_b$$

This must be the bound charge!
To verify this lets compute σ_b directly.

$$\sigma_b = \hat{n}\cdot\vec{P} = \hat{r}\cdot\varepsilon_0\chi_e\vec{E}(R^-) = \frac{\lambda}{2\pi R}\frac{\varepsilon_0\chi_e}{\varepsilon}$$

Now $\frac{\varepsilon_0}{\varepsilon} = \frac{1}{K}$ and $1+\chi_e = K \Rightarrow \chi_e = K-1$

so $\sigma_b = \frac{\lambda}{2\pi R}\frac{K-1}{K} = \frac{\lambda}{2\pi R}\left(1 - \frac{1}{K}\right)$ agrees with what we found from the jump in $\vec{E}$.

* As long as one stays <u>inside</u> the dielectric $\vec{D} = \varepsilon \vec{E}$

$$\Rightarrow \quad \vec{\nabla} \times \vec{D} = \varepsilon \vec{\nabla} \times \vec{E} = 0$$

$\Rightarrow$ $\left.\begin{array}{l} \vec{\nabla} \cdot \vec{D} = \rho_f \\ \vec{\nabla} \times \vec{D} = 0 \end{array}\right\}$ looks just like $\left\{\begin{array}{l} \vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \\ \vec{\nabla} \times \vec{E} = 0 \end{array}\right.$

except $\vec{D}$ takes place of $\vec{E}$, free charge ρ_f takes place of total charge $\rho = \rho_f + \rho_b$, and ε_0 is replaced by 1.

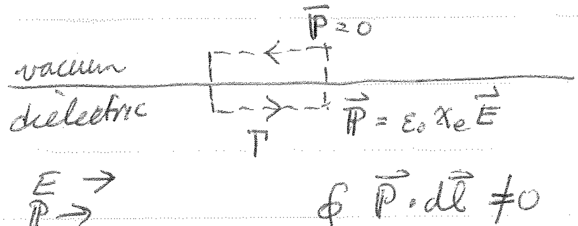
Can use all old methods to calculate $\vec{E}$ given ρ, now to calculate $\vec{D}$ given ρ_f. After we find $\vec{D}$, we can find $\vec{E} = \vec{D}/\varepsilon$

Only problem is at <u>interface</u> between dielectric + vacuum or between two dielectrics, where ε is not constant in space.

Then $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times (\varepsilon \vec{E}) \neq 0$ in <u>general</u> (for some problems $\nabla \times D = 0$, for some problems $\nabla \times D \neq 0$)

ε ← changes value as cross interface

ex:



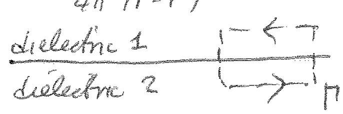
$$\oint_\Gamma \vec{\mathbb{P}} \cdot d\vec{l} \neq 0 \quad \text{as } \vec{\mathbb{P}} = 0 \text{ outside dielectric, } \vec{\mathbb{P}} \neq 0 \text{ inside dielectric}$$

$$\Downarrow$$

$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times (\varepsilon_0 \vec{E} + \vec{\mathbb{P}}) = \vec{\nabla} \times \vec{\mathbb{P}} \neq 0$$

$\Rightarrow$ cannot in general write $\vec{D}(\vec{r}) = \int d^3r' \, \frac{\rho_f(\vec{r}')}{4\pi |\vec{r} - \vec{r}'|}$ and $\vec{D}(\vec{r}) \neq -\vec{\nabla} V'$ for some V'

<u>Need boundary conditions</u>

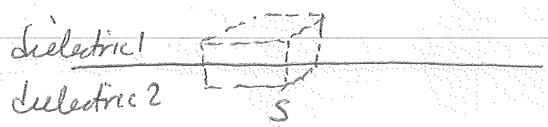


(either dielectric might be the vacuum)

$$\vec{\nabla} \times \vec{E} = 0 \text{ remains true} \Rightarrow \oint_\Gamma d\vec{l} \cdot \vec{E} = 0$$

$$\text{shrink width of loop to zero} \Rightarrow d\vec{l} \cdot \left(\vec{E}_{above} - \vec{E}_{below} \right) = 0$$

$\Rightarrow$ $\boxed{\text{Tangential part of } \vec{E} \text{ continuous}}$



$\vec{\nabla}\cdot\vec{D} = \rho_f$ depends only on "free" charge not on bound charges in dielectric

$\oint \vec{D}\cdot d\vec{a} = Q^f_{enc}$ shrink width of box to zero

$\Rightarrow da\,\hat{n}\cdot(\vec{D}_{above} - \vec{D}_{below}) = da\,\sigma_f$

$\hat{n}\cdot(\vec{D}_{above} - \vec{D}_{below}) = \sigma_f$

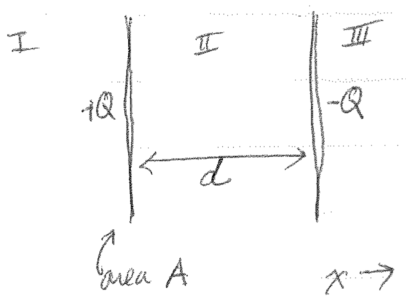
free surface charge at interface

outward normal

jump in <u>normal</u> component of $\vec{D}$ given by σ_f.

if $\sigma_f = 0$ at interface, <u>normal</u> component of $\vec{D}$ is continuous.

ex: <u>parallel plate capacitor:</u> free charge $+Q$, $-Q$ on left + right plates



<u>if vacuum inbetween</u>, $\vec{E} = 0$ in I and III,

$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{x}$ in II, $\sigma = \frac{Q}{A}$

potential drop

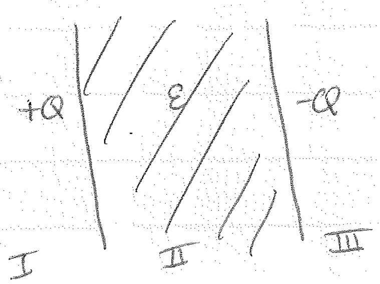
$$\Delta V = V_+ - V_- = -\int_{right}^{left} \vec{E}\cdot d\vec{l} = -\int_{d}^{0} \frac{\sigma}{\varepsilon_0}\,dx$$

$$\Delta V = \frac{\sigma d}{\varepsilon_0} = \frac{Qd}{A\varepsilon_0}$$

capacitance $C = \frac{Q}{\Delta V} = \frac{A\varepsilon_0}{d}$

if dielectric inbetween

$\sigma_f = \begin{cases} +\sigma = Q/A & \text{on left} \\ -\sigma = -Q/A & \text{on right} \end{cases}$



$\vec{D} = \varepsilon \vec{E}$ in II

$\vec{D} = \varepsilon_0 \vec{E}$ in I and III ($\varepsilon = \varepsilon_0$ in vacuum)

Solve for $\vec{D}$ in terms of free charge σ_f, just like solved for $\vec{E}$ in terms of σ^{total} for vacuum inbetween

By symmetry $\vec{D}(\vec{r}) = D(x)\hat{x}$ $\Big\{$ just like $\vec{E}(\vec{r})$ when vacuum inbetween

$\oint \vec{D}\cdot d\vec{a} = Q_{encl}^{free} \Rightarrow \vec{D}(\vec{r}) = 0$ in I and III
$= \sigma\hat{x}$ in II $\Big\}$ has correct jump in $\vec{D}\cdot\hat{n}$ as cross planes

$\Rightarrow \vec{E} = 0$ in I and III

$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\sigma}{\varepsilon}\hat{x}$ in II — compare to vacuum case, $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{x}$

$= \frac{\sigma}{\varepsilon_0 K}\hat{x}$ putting dielectric inbetween reduces electric field by factor $\frac{\varepsilon_0}{\varepsilon} = \frac{1}{K}$

potential drop $\Delta V = V_+ - V_- = -\int_d^0 E\,dx = \frac{\sigma}{\varepsilon}d$

$= \frac{Q}{A\varepsilon}d$

capacitance $C = \frac{Q}{\Delta V} = \frac{A\varepsilon}{d}$ compared to vacuum case $C = \frac{A\varepsilon_0}{d}$
capacitance increases by factor $\frac{\varepsilon}{\varepsilon_0} = K$

Polarization in dielectric: $\vec{P} = \varepsilon_0\chi_e\vec{E} = \varepsilon_0\chi_e\frac{\sigma}{\varepsilon}\hat{x}$

bound charge: $\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$ as $\vec{P}$ constant

$\sigma_b = \hat{n}\cdot\vec{P} = \frac{\varepsilon_0\chi_e\sigma}{\varepsilon}$ on right plane ($\hat{n} = \hat{x}$)

$= -\frac{\varepsilon_0\chi_e\sigma}{\varepsilon}$ on left plane ($\hat{n} = -\hat{x}$)

$\frac{\varepsilon_0}{\varepsilon}\chi_e = \frac{\chi_e}{K} = \frac{\chi_e}{1+\chi_e}$

$\sigma_b = \dfrac{\varepsilon_0 \chi_e \sigma}{\varepsilon}$ on right $\qquad$ on right $\sigma_f = -\sigma$

$\quad\;\; \dfrac{-\varepsilon_0 \chi_e \sigma}{\varepsilon}$ on left $\qquad$ on left $\sigma_f = \sigma$

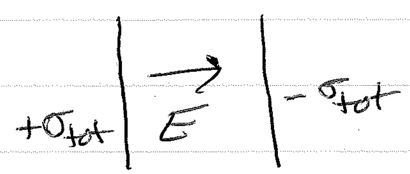
$\Rightarrow \sigma_b = \dfrac{-\varepsilon_0 \chi_e}{\varepsilon}\sigma_f = \dfrac{-\chi_e}{1+\chi_e}\sigma_f$ on both plates

Total charge on plates is

$$\sigma_{tot} = \sigma_f + \sigma_b = \sigma_f\left(1 - \frac{\chi_e}{1+\chi_e}\right) = \frac{\sigma_f}{1+\chi_e} = \frac{\sigma_f}{K}$$

total charge on plate is reduced from σ_f by factor $1/K$

We can now compute $\vec{E}$ between the plates directly using σ_{tot}



$$\vec{E} = \frac{\sigma_{tot}}{\varepsilon_0}\hat{x} = \frac{\sigma_f}{\varepsilon_0 K}\hat{x} = \frac{\sigma_f}{\varepsilon}\hat{x} = \frac{\sigma}{\varepsilon}\hat{x}$$

since $K = \varepsilon/\varepsilon_0$ $\qquad$ same as found before

Energy stored in capacitor = work done to add the free charge to the plates

$$W = \tfrac{1}{2}C(\Delta V)^2 = \tfrac{1}{2}\frac{Q^2}{C}$$

$$W = \tfrac{1}{2}C(\Delta V)^2 = \tfrac{1}{2}\left(\frac{A\varepsilon}{d}\right)\left(\frac{\sigma}{\varepsilon}d\right)^2 = \tfrac{1}{2}(Ad)\sigma\cdot\frac{\sigma}{\varepsilon}$$

or $\dfrac{1}{2}\dfrac{Q^2}{C} = \dfrac{1}{2}\dfrac{Q^2 d}{A\varepsilon} = \dfrac{1}{2}\dfrac{\sigma^2 A^2 d}{A\varepsilon} = \dfrac{1}{2}(Ad)\sigma\cdot\dfrac{\sigma}{\varepsilon}$ as before

$W = \tfrac{1}{2}(\text{vol})\,E\,D$ $\qquad$ as $\qquad E = \dfrac{\sigma}{\varepsilon}$, $D = \sigma$

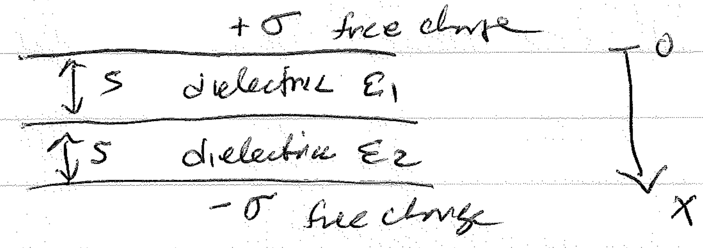
one can show (see Griffiths 4.4.3) that

$$W = \frac{1}{2} \int d^3r \; \vec{D} \cdot \vec{E}$$

holds more generally for linear dielectrics where W is the work we have to do to move the free charges - it does not include the work done to assemble the bound charges that make up the dielectric itself.

Examples

4.18



a) find $\vec{D}$

D is determined by the free charge only – doesn't depend on dielectrics

By symmetry we expect $\vec{D}(\vec{r}) = D(x)\,\hat{x}$

$$\vec{D} = \begin{cases} 0 & x<0,\ 2s<x \\ \sigma\,\hat{x} & 0<x<2s \end{cases}$$

b) find $\vec{E}$ $\qquad \vec{E} = \vec{D}/\varepsilon$ in dielectric, $\vec{E} = \vec{D}/\varepsilon_0$ in vacuum

$$\vec{E} = \begin{cases} 0 & x<0,\ 2s<x \\ \dfrac{\sigma}{\varepsilon_1}\,\hat{x} & 0<x<s \quad \text{in dielectric 1} \\ \dfrac{\sigma}{\varepsilon_2}\,\hat{x} & s<x<2s \quad \text{in dielectric 2} \end{cases}$$

c) find polarization $\vec{P}$

$\vec{P} = \varepsilon_0 \chi_e \vec{E}$ $\qquad$ use $\varepsilon = \varepsilon_0(1+\chi_e)$

so $\chi_e = \dfrac{\varepsilon}{\varepsilon_0} - 1$

$\varepsilon_0 \chi_e = \varepsilon - \varepsilon_0$

$$\vec{P} = \begin{cases} 0 & x<0,\ 2s<x \\ \dfrac{(\varepsilon_1-\varepsilon_0)}{\varepsilon_1}\sigma\,\hat{x} & 0<x<s \quad \text{in dielectric 1} \\ \dfrac{(\varepsilon_2-\varepsilon_0)}{\varepsilon_2}\sigma\,\hat{x} & s<x<2s \quad \text{in dielectric 2} \end{cases}$$

d) find potential difference $\Delta V = V_+ - V_- = -\int_{2s}^{0} dx\, E(x)$

$$= \int_0^{2s} dx\, E(x)$$

$$\Delta V = \int_0^{s} E_1\, dx + \int_s^{2s} E_2\, dx$$

$$= \frac{s\sigma}{\varepsilon_1} + \frac{s\sigma}{\varepsilon_2} = s\sigma\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)$$

we can now also compute the capacitance

$$C = \frac{Q}{\Delta V} = \frac{A\sigma}{s\sigma\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)} = \frac{A}{s}\,\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}$$

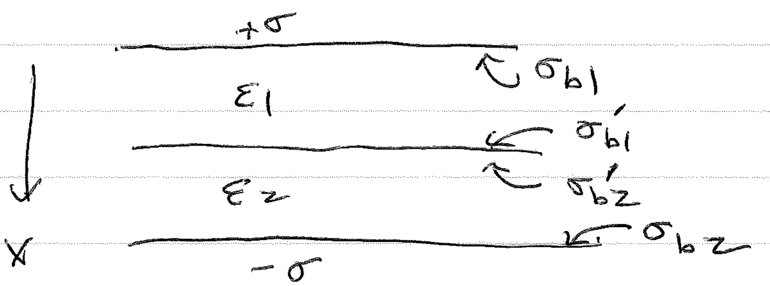
Note: if we write $C_1 = \frac{A\varepsilon_1}{s}$ and $C_2 = \frac{A\varepsilon_2}{s}$

then $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$ (the capacitors add like resistors in parallel)

e) Find the bound charge

$$\rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}} = 0$$ since $\vec{\mathbb{P}}$ is uniform in the two dielectrics

$$\sigma_b = \hat{n}\cdot\vec{\mathbb{P}}$$



$$\sigma_{b1} = -\hat{x}\cdot\vec{\mathbb{P}}_1 = -\frac{(\varepsilon_1-\varepsilon_0)}{\varepsilon_1}\sigma$$

$$\sigma_{b1}' = +\hat{x}\cdot\vec{\mathbb{P}}_1 = \frac{(\varepsilon_1-\varepsilon_0)}{\varepsilon_1}\sigma$$

$$\sigma_{b2}' = -\hat{x}\cdot\vec{\mathbb{P}}_2 = -\frac{(\varepsilon_2-\varepsilon_0)}{\varepsilon_2}\sigma$$

$$\sigma_{b2} = +\hat{x}\cdot\vec{\mathbb{P}}_2 = \frac{(\varepsilon_2-\varepsilon_0)}{\varepsilon_2}\sigma$$

f) Now find $\vec{E}$ directly from the total charge $\sigma_{tot} = \sigma_f + \sigma_b$

First we find the total surface charge at each interface

I

$\sigma_a = \sigma + \sigma_{b1}$

II $\quad \varepsilon_1$

$\sigma_b = \sigma_{b1}' + \sigma_{b2}'$

III $\quad \varepsilon_2$

$\sigma_c = -\sigma + \sigma_{b2}$

$x \downarrow$ $\quad$ IV

$$\sigma_a = \sigma - \left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1}\right)\sigma = \sigma\left(1 - \left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1}\right)\right) = \sigma \frac{\varepsilon_0}{\varepsilon_1} = \frac{\sigma}{K_1}$$

$$\sigma_b = \left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1}\right)\sigma - \left(\frac{\varepsilon_2 - \varepsilon_0}{\varepsilon_2}\right)\sigma = \left(1 - \frac{1}{K_1}\right)\sigma - \left(1 - \frac{1}{K_2}\right)\sigma = \left(\frac{1}{K_2} - \frac{1}{K_1}\right)\sigma$$

$$\sigma_c = -\sigma + \left(\frac{\varepsilon_2 - \varepsilon_0}{\varepsilon_2}\right)\sigma = \sigma\left(-1 + 1 - \frac{1}{K_2}\right) = -\frac{\sigma}{K_2}$$

Now we can compute the electric fields from $\sigma_a, \sigma_b, \sigma_c$ separately, then use superposition to get the total field

$$\vec{E}_a = \begin{cases} -\dfrac{\sigma_a}{2\varepsilon_0}\hat{x} & \text{in I} \\ +\dfrac{\sigma_a}{2\varepsilon_0}\hat{x} & \text{in II, III, IV} \end{cases}$$

$$\vec{E}_b = \begin{cases} -\dfrac{\sigma_b}{2\varepsilon_0}\hat{x} & \text{in I, II} \\ +\dfrac{\sigma_b}{2\varepsilon_0}\hat{x} & \text{in III, IV} \end{cases}$$

$$\vec{E}_c = \begin{cases} -\dfrac{\sigma_c}{2\varepsilon_0}\hat{x} & \text{in I, II, III} \\ +\dfrac{\sigma_c}{2\varepsilon_0}\hat{x} & \text{in IV} \end{cases}$$

by superposition $\vec{E} = \vec{E}_a + \vec{E}_b + \vec{E}_c$

$$\vec{E}_I = \left(-\frac{\sigma_a}{2\varepsilon_0} - \frac{\sigma_b}{2\varepsilon_0} - \frac{\sigma_c}{2\varepsilon_0}\right)\hat{x} = \frac{-\hat{x}}{2\varepsilon}\left(\frac{\sigma}{K_1} + \left(\frac{\sigma}{K_2} - \frac{\sigma}{K_1}\right) - \frac{\sigma}{K_2}\right)$$

$$= 0$$

$$\vec{E}_{IV} = \left(\frac{\sigma_a}{2\varepsilon_0} + \frac{\sigma_b}{2\varepsilon_0} + \frac{\sigma_c}{2\varepsilon_0}\right)\hat{x} = -\vec{E}_I = 0$$

$$\vec{E}_{II} = \left(\frac{\sigma_a}{2\varepsilon_0} - \frac{\sigma_b}{2\varepsilon_0} - \frac{\sigma_c}{2\varepsilon}\right)\hat{x} = \frac{\hat{x}}{2\varepsilon_0}\left(\frac{\sigma}{K_1} - \left(\frac{\sigma}{K_2} - \frac{\sigma}{K_1}\right) + \frac{\sigma}{K_2}\right)$$

$$= \frac{\hat{x}}{2\varepsilon_0}\left(\frac{2\sigma}{K_1}\right) = \frac{\sigma}{\varepsilon_0 K_1}\hat{x} = \frac{\sigma}{\varepsilon_1}\hat{x}$$

$$\vec{E}_{III} = \left(\frac{\sigma_a}{2\varepsilon_0} + \frac{\sigma_b}{2\varepsilon_0} - \frac{\sigma_c}{2\varepsilon_0}\right)\hat{x} = \frac{\hat{x}}{2\varepsilon_0}\left(\frac{\sigma}{K_1} + \left(\frac{\sigma}{K_2} - \frac{\sigma}{K_1}\right) + \frac{\sigma}{K_2}\right)$$

$$= \frac{\hat{x}}{2\varepsilon_0}\cdot\left(\frac{2\sigma}{K_2}\right) = \frac{\sigma}{\varepsilon_0 K_2}\hat{x} = \frac{\sigma}{\varepsilon_2}\hat{x}$$

$$\vec{E} = \begin{cases} 0 & x < 0, \quad 2s < x \\ \frac{\sigma}{\varepsilon_1}\hat{x} & 0 < x < s \quad \text{in dielectric 1} \\ \frac{\sigma}{\varepsilon_2}\hat{x} & s < x < 2s \quad \text{in dielectric 2} \end{cases}$$

same result as in part (b)