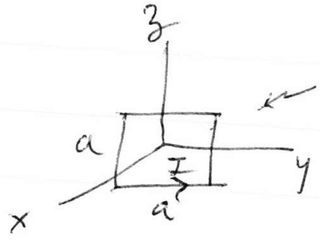
5.4 $\vec{B} = kz\,\hat{x}$



← square loop in yz plane side length

I flows counter clockwise

$$\vec{F} = \oint dl(\vec{I}\times\vec{B}) = I\int_{-\frac{a}{2}}^{\frac{a}{2}} dy\ \hat{y}\times\left(-k\frac{a}{2}\hat{x}\right) + I\int_{-\frac{a}{2}}^{a/2} dz\ \hat{z}\times(kz\,\hat{x})$$

bottom   right side

$$-I\int_{-\frac{a}{2}}^{\frac{a}{2}} dy\ \hat{y}\times\left(k\frac{a}{2}\hat{x}\right) - I\int_{-\frac{a}{2}}^{\frac{a}{2}} dz\ \hat{z}\times(kz\,\hat{x})$$

top   left side

Contributions from left side and right side cancel

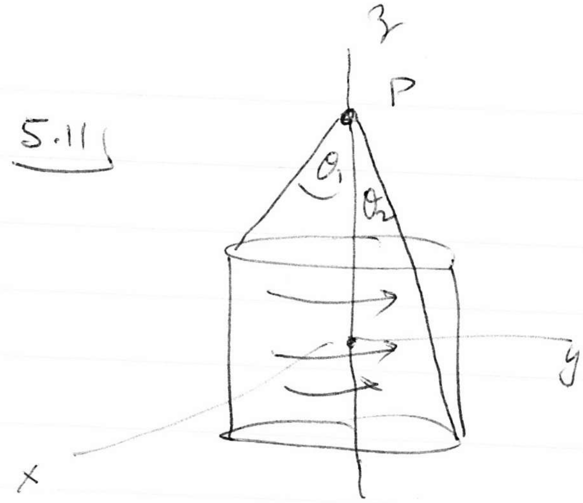
Contributions from top and bottom add

use $\hat{y}\times\hat{x} = -\hat{z}$

$$\vec{F} = -Ik\frac{a}{2}(-\hat{z})\int_{-\frac{a}{2}}^{\frac{a}{2}} dy - Ik\frac{a}{2}(-\hat{z})\int_{-\frac{a}{2}}^{a/2} dy$$

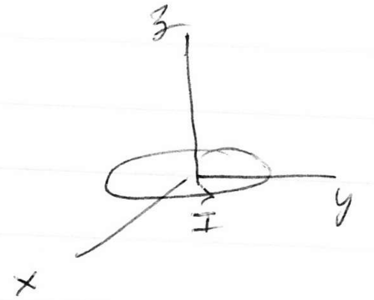
$$\boxed{\vec{F} = Ika^2\hat{z}}$$

5.11



Find $\vec{B}$ at point P on z axis from a solenoid of finite length L
solenoid has radius R, N turns/length of wire with current I.

For a single circular loop at $z=0$ we found for field at $z\hat{z}$



$$\vec{B}(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{z}$$

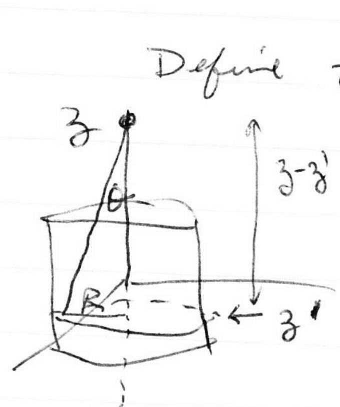
For a loop at height z' we would have

$$\vec{B}(z) = \frac{\mu_0 I}{2}\frac{R^2}{[R^2+(z-z')^2]^{3/2}}\hat{z}$$

Field from solenoid of finite length is obtained by just adding up fields from circular loops at $z = -L/2$ to $L/2$

$$\vec{B}(z) = \frac{\mu_0 R^2 \hat{z}}{2} IN \int_{-\frac{L}{2}}^{\frac{L}{2}} dz' \frac{1}{[R^2+(z-z')^2]^{3/2}}$$

Define $\tan\theta = \frac{R}{(z-z')}$ $\Rightarrow$ $z'-z = \frac{-R}{\tan\theta}$



$$dz' = \frac{R\,d\theta}{\sin^2\theta}$$

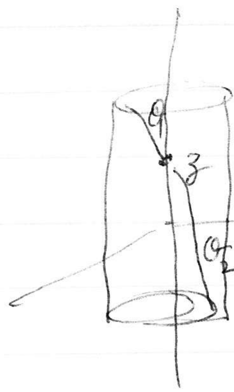
$$\frac{1}{(R^2+(z-z')^2)^{1/2}} = \frac{\sin\theta}{R}$$

$$\vec{B}(z) = \frac{\mu_0 R^2 \hat{z}}{2} IN \int_{\theta_2}^{\theta_1} d\theta \, \frac{R}{\sin^2\theta} \quad \frac{\sin^3\theta}{R^3}$$

$$= \frac{\mu_0 \hat{z}}{2} IN \int_{\theta_2}^{\theta_1} d\theta \, \sin\theta$$

$$= \frac{\mu_0 \hat{z}}{2} IN \left(-\cos\theta\right)\Big|_{\theta_2}^{\theta_1}$$

$$\boxed{\vec{B}(z) = \frac{\mu_0 IN}{2} \left(\cos\theta_2 - \cos\theta_1\right) \hat{z}}$$



$\theta_1 \to \pi$ as $L \to \infty$

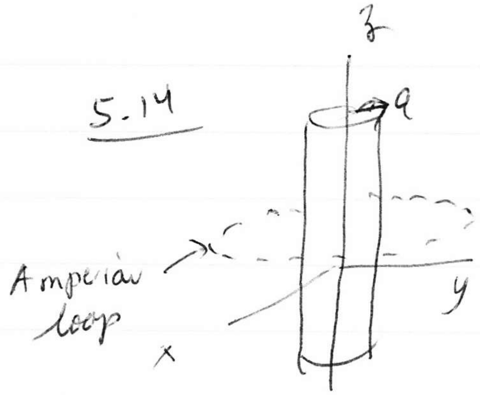
$\theta_2 \to 0$ as $L \to \infty$

For an infinite soleroid $\theta_2 = 0$, $\theta_1 = \pi$

$$\vec{B}(z) = \frac{\mu_0 IN}{2} \left(1 - (-1)\right)\hat{z} = \mu_0 IN \hat{z}$$

correct answer as $L \to \infty$

5.14



current down wire — total current is I
radius a

By symmetry we know $\vec{B}(\vec{r}) = B(r)\hat{\varphi}$

Evaluate Amperes law on circular loop of radius r

$$\oint_{\Gamma} d\vec{l}\cdot\vec{B} = \int_0^{2\pi} d\varphi\, r\,\hat{\varphi}\cdot B(r)\hat{\varphi} = 2\pi r B(r) = \mu_0 I_{encl}$$

a) If current flows uniformly distributed on outer surface of wire then

$$I_{encl} = 0 \quad r<a \qquad I_{encl} = I \quad r>a \quad \Rightarrow \quad \vec{B}(\vec{r}) = \begin{cases} 0 & r<a \\ \dfrac{\mu_0 I}{2\pi r}\hat{\varphi} & r>a \end{cases}$$

b) current distributed $\vec{J} = kr\hat{z}$ $\quad$ r is cylindrical radial distance

we must have $\underbrace{\int_0^a dr \int_0^{2\pi} d\varphi\, r}_{\text{area integral in polar coordinates}}\ \vec{J}\cdot\hat{z} = I$ total current

$$= 2\pi \int_0^a dr\, r\,(kr) = 2\pi k \int_0^a dr\, r^2 = \frac{2}{3}\pi k a^3 = I$$

so $\quad k = \dfrac{3}{2}\dfrac{I}{\pi a^3}$

Now for $r > a$, $I_{encl} = I$

for $r < a$, $I_{encl} = \int_0^r dr' \int_0^{2\pi} d\varphi \, r' \, k r' = 2\pi k \int_0^r dr' (r')^2 = \frac{2\pi k}{3} r^3$

$$= \frac{2\pi}{3} \left(\frac{3}{2} \frac{I}{\pi a^3} \right) r^3 = I \left(\frac{r}{a} \right)^3$$

$$\vec{B}(\vec{r}) = \begin{cases} \frac{\mu_0 I}{2\pi r} \left(\frac{r^3}{a^3} \right) \hat{\varphi} = \frac{\mu_0 I r^2}{2\pi a^3} \hat{\varphi} & r < a \\ \frac{\mu_0 I}{2\pi r} \hat{\varphi} & r > a \end{cases}$$

5.25 | $\vec{B}$ is uniform

Show that $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$ is a vector potential for $\vec{B}$

$$\vec{\nabla}\times\vec{A} = -\frac{1}{2}\vec{\nabla}\times(\vec{r}\times\vec{B})$$

$$(\vec{\nabla}\times\vec{A})_i = -\frac{1}{2}\varepsilon_{ijk}\partial_j\varepsilon_{klm}r_l B_m$$

$$= -\frac{1}{2}\varepsilon_{kij}\varepsilon_{klm}\partial_j r_l B_m$$

use $\partial_j r_l B_m = B_m \partial_j r_l = B_m \delta_{jl}$

$$= -\frac{1}{2}\left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right] B_m \delta_{jl}$$

$$= -\frac{1}{2}\left[\delta_{ij}B_j - \delta_{jj}B_i\right] \qquad \sum_{j=1}^{3}\delta_{jj} = 3$$

$$= -\frac{1}{2}\left[B_i - 3B_i\right] = -\frac{1}{2}\left[-2B_i\right] = B_i$$

So $\vec{\nabla}\times\vec{A} = \vec{B}$

No, the $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$ is not unique, can always add to it any $\vec{\nabla}\lambda$ for any scalar λ

Ex: for $\vec{B} = B\hat{z}$ then $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B}) = -\frac{1}{2}yB\hat{x} + \frac{1}{2}xB\hat{y}$

choose $\lambda = \frac{1}{2}xyB$ then $\vec{\nabla}\lambda = \frac{1}{2}yB\hat{x} + \frac{1}{2}xB\hat{y}$

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda = -\frac{1}{2}yB\hat{x} + \frac{1}{2}xB\hat{y} + \frac{1}{2}yB\hat{x} + \frac{1}{2}xB\hat{y} = xB\hat{y}$$

if choose $\lambda = -\frac{1}{2}xyB$ then $\vec{A}' = yB\hat{x}$