Statics Review

charge density $\rho(\vec{r})$

$$\int_{\text{volume } V} d^3r\, \rho(\vec{r}) = Q$$ total charge inside volume V

current density $\vec{j}(\vec{r})$

$$\int_{\text{surface } S} d\vec{a} \cdot \vec{j}(\vec{r}) = I$$ total current (charge per unit time) flowing through surface S

local charge conservation

$$\frac{\partial \rho(\vec{r})}{\partial t} + \vec{\nabla} \cdot \vec{j}(\vec{r}) = 0$$

in electrostatics, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla} \cdot \vec{j}(\vec{r}) = 0$ is key condition for magnetostatics

– also $\frac{\partial \vec{j}}{\partial t} = 0$

Maxwell's Equations

	electrostatics	magnetostatics
integral form:	Gauss $\oint_{\text{surface } S} \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$	$\oint_S \vec{B} \cdot d\vec{a} = 0$
	$\oint_{\text{curve } C} \vec{E} \cdot d\vec{l} = 0$	Ampere $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$
differential form:	$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$	$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$
	$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$	$\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \vec{j}(\vec{r})$

Now move onto dynamics

Faraday's Law of Induction

The electromotive force. Suppose we have a closed wire loop. If there is a current flowing around the loop, then there must be an electromagnetic force pushing the charge around the loop. Let $\vec{f} = \vec{F}/q$ be the force per unit charge. $\vec{f} = \vec{E} + \vec{v} \times \vec{B}$. The electromotive force "emf" is the integral of $\vec{f}$ around the loop

$$\text{emf: } \mathcal{E} = \oint_C d\vec{l} \cdot \vec{f} = \oint_C d\vec{l} \cdot \vec{E} + \oint_C d\vec{l} \cdot (\vec{v} \times \vec{B})$$

$\vec{v}$ is the total velocity of the charge.

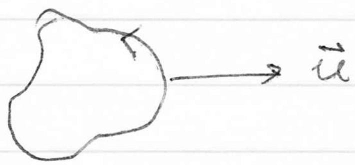
We will consider two cases:

① We are in an electro- and magneto-static situation where $\vec{E}$ and $\vec{B}$ do not vary with time. But we will see that if the loop is moving, then $\mathcal{E} \neq 0$ due to the $\vec{B}$ field part of $\vec{f}$. This is called the "motional" emf.

② The loop is not moving but $\vec{B}$ is changing in time. Relativity will then imply that there must be an $\mathcal{E} \neq 0$ around the loop due to an induced $\vec{E}$ field. This will give Faraday's Law.

Case ① Since we are in an electrostatic situation, we know $\vec{\nabla}\times\vec{E}=0$ so $\oint_C d\vec{l}\cdot\vec{E}=0$ and only the magnetic part can give rise to a non-zero $\mathcal{E}$.

Let ~~one~~ us assume the loop has a rigid shape and is moving with a constant velocity $\vec{u}$ in a magnet field $\vec{B}(\vec{r})$ that is constant in time, but is spatially non-uniform.



C

Let $\vec{R}$ denote the center of mass of the loop, and $\vec{r}'$ the relative coordinate with respect to the center of mass. $\vec{r}=\vec{R}+\vec{r}'$ is then the position of the charge as it goes around the loop. When we integrate around the loop we will use the coordinate $\vec{r}'$ which does not change even as the loop moves. $d\vec{l}=d\vec{r}'$

The velocity of the charge is then

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{d\vec{R}}{dt}+\frac{d\vec{r}'}{dt}=\vec{u}+\frac{d\vec{r}'}{dt}$$

$$\mathcal{E}=\oint_c d\vec{l}\cdot(\vec{v}\times\vec{B})=\oint_C d\vec{r}'\cdot\left[\left(\vec{u}+\frac{d\vec{r}'}{dt}\right)\times\vec{B}\right]$$

$\vec{r}'$ lying on curve C

$$=\oint_C d\vec{r}'\cdot\left(\frac{d\vec{r}'}{dt}\times\vec{B}\right)+\oint_C d\vec{r}'\cdot(\vec{u}\times\vec{B})$$

The first term can be written as

$$\oint_C d\vec{r}' \cdot \left(\frac{d\vec{r}'}{dt} \times \vec{B} \right) = \oint_C dt \, \frac{d\vec{r}'}{dt} \cdot \left(\frac{d\vec{r}'}{dt} \times \vec{B} \right) = 0$$

since $\frac{d\vec{r}'}{dt} \times \vec{B}$ must be $\perp$ to $\frac{d\vec{r}'}{dt}$, ie $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ for any vectors $\vec{a}$ and $\vec{b}$.

Thus the emf is due solely to the motion of the loop

$$\mathcal{E} = \oint_C d\vec{r}' \cdot (\vec{u} \times \vec{B})$$

vanishes if the loop is not moving, ie if $\vec{u} = 0$.

Next rewrite the above using Stoke's law of vector calculus

$$\mathcal{E} = \oint_C d\vec{r}' \cdot (\vec{u} \times \vec{B}) = \int_S d^2 r' \, \hat{n} \cdot \vec{\nabla} \times (\vec{u} \times \vec{B})$$

S ← surface bounded by loop C
$\hat{n}$ is outward pointing normal vector

~~use the vector identity~~

use vector identity (see cover Griffiths)

$$\vec{\nabla} \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \vec{\nabla})\vec{a} - (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{a}(\vec{\nabla} \cdot \vec{b}) - \vec{b}(\vec{\nabla} \cdot \vec{a})$$

apply with $\vec{b} = \vec{B}$, $\vec{a} = \vec{u}$ constant

$$\vec{\nabla} \times (\vec{u} \times \vec{B}) = -(\vec{u} \cdot \vec{\nabla})\vec{B} + \vec{u} \underbrace{(\vec{\nabla} \cdot \vec{B})}_{=0} = -(\vec{u} \cdot \vec{\nabla})\vec{B}$$

$$\mathcal{E} = -\int_S d^2 r' \, \hat{n} \cdot \left[(\vec{u} \cdot \vec{\nabla}) \vec{B} \right]$$

Finally, $(\vec{u}\cdot\vec{\nabla})\vec{B} = \frac{d\vec{B}}{dt}$

as we integrate over surface S bounded by the loop, we evaluate $\vec{B}(\underset{\uparrow}{\vec{R}(t)}+\underset{\uparrow}{\vec{r}'})$

center of mass coordinate — relative coordinate indep of time t

$$\frac{d\vec{B}}{dt} = \sum_i \frac{\partial \vec{B}}{\partial x_i}\frac{dR_i}{dt} = \sum_i u_i \frac{\partial \vec{B}}{\partial x_i} = (\vec{u}\cdot\vec{\nabla})\vec{B}$$

(assumes $\vec{B}(\vec{r})$ is not itself changing in time).

$$\mathcal{E} = -\oint_S d^2r'\, \hat{n}\cdot\frac{d\vec{B}}{dt} = -\frac{d}{dt}\oint_S d^2r'\,\hat{n}\cdot\vec{B}$$

$$= -\frac{d\Phi}{dt}$$ where $\Phi = \int_S d\vec{a}\cdot\vec{B}$ is magnetic flux through surface

So in case ① of loop moving in time independent $\vec{B}$ we get

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$ Faraday's Law of Induction

Now to case ② go to rest frame of moving loop in case ①. Now loop is not moving (in this rest frame) but $\vec{B}$ is now varying in time, ie if sit at fixed position $\vec{r}$ in the loops rest frame then see $\vec{B}$ changing in time since in this frame $\vec{B}$ field is moving with velocity $-\vec{u}$.

By relativity we expect that we must still find

$$\mathcal{E} = -\frac{d\Phi}{dt}$$ in the loop rest frame

(otherwise we would have a way to know the absolute state of motion of the loop).

But now the "motional" part of the emf must be zero since the loop is not moving. So the emf <u>must</u> therefore be due to an $\vec{E}$ field

$$\mathcal{E} = \oint_C d\vec{l} \cdot \vec{E} \neq 0$$

so unlike electrostatics where $\oint_C d\vec{l} \cdot \vec{E} = 0$ always, once we have a time varying magnetic field we have

$$\mathcal{E} = \oint_C d\vec{l} \cdot \vec{E} = -\frac{d\Phi}{dt}$$

Note: in frame of moving loop, $\mathcal{E}$ due entirely to $\vec{B}$ - there is no $\vec{E}$ present. In rest frame of loop, $\mathcal{E}$ due entirely to $\vec{E}$. $\Rightarrow$ under Lorentz transformations $\vec{B}$ and $\vec{E}$ fields can map into each other - what is $\vec{E}$ and what is $\vec{B}$ depends on the reference frame you are in!

$$\oint_C d\vec{l} \cdot \vec{E} = -\frac{d\Phi}{dt}$$

Stokes ||

$$\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{E}) = -\frac{d}{dt} \int_S d\vec{a} \cdot \vec{B} = -\int_S d\vec{a} \cdot \frac{\partial \vec{B}}{\partial t}$$

for loop that is not moving.

Must be true for any surface S

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Faraday's Law in differential form.

Extends our static law $\vec{\nabla} \times \vec{E} = 0$ to situations where $\vec{B}$ varies with time

energy stored up in configuration of current carrying loops

What is the work done to create configuration of the two loops above?

1) move loops into position, with initially $I_1 = I_2 = 0$.
   when $I_1 = I_2 = 0$, the loops exert no force on each other, so this step takes no work.

2) Now turn up current in loop 1 to I_1
   " " " " " 2 to I_2

This step costs work as follows: As I_1 and I_2 change in time, they give rise to changing magnetic flux through the loops, which gives rise to emf's around the loops, which tends to oppose the ~~change in~~ currents I_1 and I_2. To keep the currents I_1 and I_2 flowing, we have to do work against this induced emf.

emf in loop 1: $\mathcal{E}_1 = \underbrace{-L_1 \frac{dI_1}{dt}}_{\text{self inductance}} \underbrace{- M \frac{dI_2}{dt}}_{\text{mutual inductance}}$

emf in loop 2: $\mathcal{E}_2 = \overbrace{-L_2 \frac{dI_2}{dt}} \overbrace{- M \frac{dI_1}{dt}}$

~~The emf is the work done per charge, moving charge around loop.~~

~~$\mathcal{E} I$ is power dissipated by the emf~~

~~charge/time~~

The emf $\mathcal{E}_1$ and $\mathcal{E}_2$ oppose the change in I_1 and I_2. To keep I_1 and I_2 steady, we have to ~~wo~~ do work to counter this emf ~~to keep the steady current I flowing, we have to pay back in this power lost. So the total work done per unit time is~~

$$\frac{dW}{dt} = -\underbrace{\mathcal{E}_1}_{\text{work per charge in one trip around loop}} I_1 - \mathcal{E}_2 \underbrace{I_2}_{\text{charge per time traveling down wire}}$$

$$\frac{dW}{dt} = L_1 I_1 \frac{dI_1}{dt} + M I_1 \frac{dI_2}{dt} + L_2 I_2 \frac{dI_2}{dt} + M I_2 \frac{dI_1}{dt}$$

$$= \frac{1}{2} L_1 \frac{d(I_1^2)}{dt} + \frac{1}{2} L_2 \frac{d(I_2^2)}{dt} + M \frac{d(I_1 I_2)}{dt}$$

$$W = \int_0^T \frac{dW}{dt}\, dt = \underbrace{\frac{1}{2} L_1 I_1^2}_{\substack{\text{self energy}\\ \text{of loop 1}}} + \underbrace{\frac{1}{2} L_2 I_2^2}_{\substack{\text{self energy}\\ \text{of loop 2}}} + \underbrace{M I_1 I_2}_{\substack{\text{interaction energy}\\ \text{between loops 1 + 2.}}}$$

substitute in expressions for L_1, L_2, M

$$W = \frac{1}{2}\frac{\mu_0}{4\pi} I_1^2 \oint_1 \oint_1 \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r}-\vec{r}'|} + \frac{1}{2}\frac{\mu_0}{4\pi} I_2^2 \oint_2 \oint_2 \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

$$+ \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

$$\cancel{= \frac{1}{2}\frac{\mu_0}{4\pi}\left\{ \oint_1 \oint dl\, dl' \frac{\vec{I}_1 \cdot \vec{I}_1}{|\vec{r}-\vec{r}'|}\right.}$$

$$= \frac{1}{2}\frac{\mu_0}{4\pi}\left\{ \oint_1 \oint_1 dl\, dl' \frac{\vec{I}(\vec{r}) \cdot \vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_2 \oint_2 dl\, dl' \frac{\vec{I}(\vec{r}) \cdot \vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right.$$

$$\left. + 2 \oint_1 \oint_2 dl\, dl' \frac{\vec{I}(\vec{r}) \cdot \vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right\}$$

where $\vec{I}(\vec{r})$ = current at position $\vec{r}$ on either loop 1 or loop 2 (depending on where $\vec{r}$ is)

$$W = \frac{1}{2}\frac{\mu_0}{4\pi}\oint_{1+2}\oint_{1+2} \cancel{dl}\, dl\, dl' \frac{\vec{I}(r)\cdot\vec{I}(r')}{|\vec{r}-\vec{r}'|}$$

generalization to n loops: $W = \frac{1}{2}\frac{\mu_0}{4\pi}\oint_{1+2+\cdots n}\oint_{1+2+\cdots n} dl\, dl' \frac{\vec{I}(\vec{r})\cdot\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

generalization to continuous current distribution:

$$W = \frac{1}{2}\frac{\mu_0}{4\pi}\iint d^3r\, d^3r' \frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ (compare to electrostatic energy $W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\iint d^3r\, d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$)

use $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$ in Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$

$$W = \frac{1}{2}\int d^3r\, \vec{A}(\vec{r})\cdot\vec{j}(\vec{r})$$ (compare to electrostatic energy $W = \frac{1}{2}\int d^3r\, V(\vec{r})\rho(\vec{r})$)

For a steady current, ie magnetostatic situation,

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j}$$

$$\Rightarrow W = \frac{1}{2\mu_0}\int d^3r\, \vec{A}(\vec{r})\cdot\left[\vec{\nabla}\times\vec{B}(\vec{r})\right]$$ use $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

$$= \frac{1}{2\mu_0}\int d^3r\left[\vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B})\right]$$

$$= \frac{1}{2\mu_0}\int_{vol} d^3r\, B^2 - \frac{1}{2\mu_0}\oint_{surface}(\vec{A}\times\vec{B})\cdot d\vec{a}$$

as let volume ~~include~~ all of space, surface $\to\infty$, then for localized current source $\vec{j}$, the ~~sto~~ $\vec{B}$ and $\vec{A}$ will $\to 0$ as $\vec{r}\to\infty$ sufficiently fast that the surface integral will vanish.

$$\Rightarrow W = \frac{1}{2\mu_0}\int d^3r\, B^2$$ (compare to electostatic $W = \frac{1}{2}\epsilon_0\int d^3r\, E^2$)

Summary

magnetostatic energy

$$W = \frac{1}{2}\frac{\mu_0}{4\pi}\int d^3r \int d^3r' \, \frac{\vec{J}(\vec{r})\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$W = \frac{1}{2}\int d^3r \, \vec{J}(\vec{r})\cdot\vec{A}(\vec{r})$$

$$W = \frac{1}{2\mu_0}\int d^3r \, |\vec{B}(\vec{r})|^2$$

electrostatic energy

$$W = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\int d^3r \int d^3r' \, \frac{\rho(\vec{r})\,\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$W = \frac{1}{2}\int d^3r \, \rho(\vec{r})\,\Phi(\vec{r})$$

$$W = \frac{1}{2}\varepsilon_0\int d^3r \, |\vec{E}(\vec{r})|^2$$