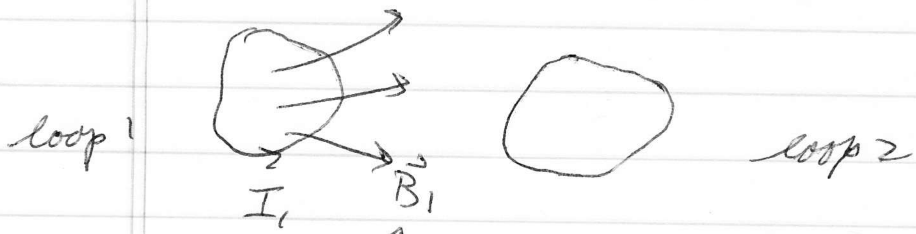
Mutual and self inductance of current loops

Mutual inductance of two loops



↑ magnetic field due to current in loop 1

What is magnetic flux Φ_2 through loop 2 due to current I_1 in loop 1.

$$\Phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \int_{S_2} (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{a}_2 = \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2$$

S_2 surface enclosed by loop 2

since $\vec{B}_1 = \vec{\nabla} \times \vec{A}_1$

In the Coulomb gauge $-\nabla^2 \vec{A} = \mu_0 \vec{j}$

$$\Rightarrow \vec{A}_1(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{j}_1(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

$$= \frac{\mu_0}{4\pi} \oint_{C_1} d\vec{l}_1 \, \frac{I_1}{|\vec{r} - \vec{r}_1|}$$ current is only along loop C_1

$$\Phi_2 = \oint_{C_2} d\vec{l}_2 \cdot \vec{A}_1(\vec{r}_2) = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1 \, I_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\equiv M_{21} I_1$$

↑ mutual inductance of loops 1 and 2

Similarly one can compute the flux Φ_1 through loop 1 due to current I_2 in loop 2. One gets

$$\Phi_1 = \frac{\mu_0}{4\pi} \oint_{C_1}\oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2 \, I_2}{|\vec{r}_1 - \vec{r}_2|} \equiv M_{12} I_2$$

we see that $M_{12} = M_{21} \equiv M = \oint_{C_1}\oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|}$

mutual inductance is determined solely by the geometry of the loops.

If we vary the current in loop 1, then the flux through loop 2 will change $\Rightarrow$ emf develops around loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

$\Rightarrow$ induced current $I_2 = \dfrac{\mathcal{E}_2}{R_2}$ $\leftarrow$ resistance of loop 2
flows in loop 2 when current in loop 1 is changed. This is the principle behind a transformer.

<u>Self inductance</u>

What is the magnetic flux through a loop due to the current flowing in that same loop?

$$\Phi = \oint_C \vec{A} \cdot d\vec{l} = \frac{\mu_0}{4\pi} \oint_C \oint_C \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r} - \vec{r}'|} I \equiv L I$$

self inductance

both integrals go over same loop

$$L = \frac{\mu_0}{4\pi} \oint_c \oint_c \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r} - \vec{r}'|}$$

inductance is measured in "henries" (H)

$$1\,H = 1 \text{ volt-sec/amp}$$

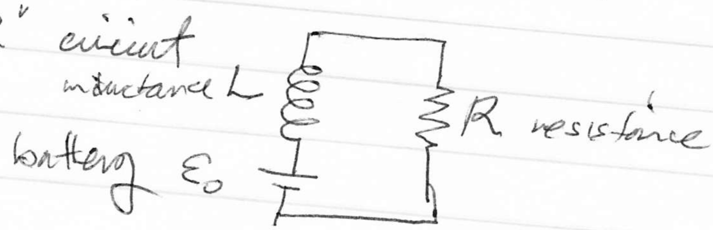
Changing I in loop changes Φ through the loop, $\Rightarrow$ creates emf around the loop

$$\mathcal{E} = -L \frac{dI}{dt} \qquad L > 0 \text{ always}$$

this induced emf $\mathcal{E}$ always acts to <u>oppose</u> the change in current – it is called the back emf.

if I counterclockwise is increased, then the induced $\mathcal{E} = -L\frac{dI}{dt}$ is negative so as to create an induced current that is clockwise, ie opposite to the direction of the increase in I.

<u>ex:</u> "LR" circuit



total emf in circuit is

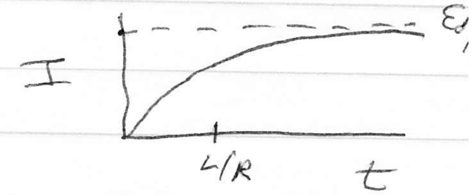
$$\underbrace{\mathcal{E}_0}_{\text{battery}} \underbrace{- L \frac{dI}{dt}}_{\text{induced}} = \underbrace{IR}_{\text{voltage drop across resistor}}$$

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR \qquad \frac{dI}{dt} = -\frac{R}{L}I + \frac{\mathcal{E}_0}{L}$$

1st order differential eqn for $I(t)$

if switch on battery at $t=0$, the solution is

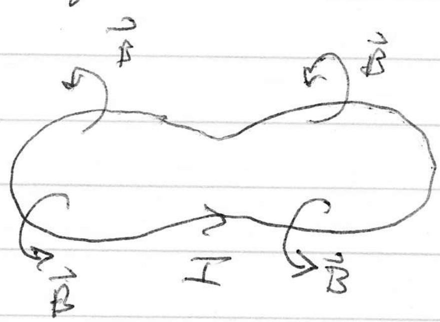
$$I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-(R/L)t}\right)$$



current increases to steady state value $\mathcal{E}_0/R$

over time scale $\tau = L/R$

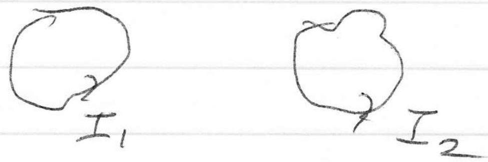
self inductance L is always positive



each segment $I \rightarrow$ generates a field $\vec{B}$ that circulates around the segment according to the right hand rule.

$\Rightarrow$ net flux is always positive for counter clockwise (ie positive) current.

Back to energy of two interacting current loops



Method ①

i) move loops into position with $I_1 = I_2 = 0$ — costs no work

ii) Then turn up currents to final values I_1, I_2

$$\mathcal{E}_1 = -L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt}$$

$$\mathcal{E}_2 = -L_2 \frac{dI_1}{dt} - M \frac{dI_1}{dt}$$

$$\frac{dW}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2 = L_1 I_1 \frac{dI_1}{dt} + L_2 I_2 \frac{dI_2}{dt} + M I_1 \frac{dI_2}{dt} + M I_2 \frac{dI_1}{dt}$$

$$\frac{dW}{dt} = \frac{1}{2} L_1 \frac{d(I_1^2)}{dt} + \frac{1}{2} L_2 \frac{d(I_2^2)}{dt} + M \frac{d(I_1 I_2)}{dt}$$

$$W = \int_0^T \frac{dW}{dt} dt' = \underbrace{\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2}_{\text{self energies of loop1 and loop2}} + \underbrace{M I_1 I_2}_{\text{interaction energy}}$$

Method ②

i) Turn on currents I_1 and I_2 while loops separated infinitely apart.

ii) Then move loops together into final positions

Work done in step (i) is just the self energies

$$\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2$$

Work done in (ii) involves the force acting between the two loops that we must overcome to push them together

Start off more generally: What is total Lorentz force on a current distribution 2 due to the magnetic field from another current distribution 1.

$$\vec{F}_2 = \int d^3r_2 \; \vec{J}_2(\vec{r}_2) \times \vec{B}_1(\vec{r}_2)$$

field from $\vec{J}(\vec{r}_1)$

(to check, consider $\vec{J}_2(\vec{r}) = \sum_{\substack{i \text{ in} \\ \text{distr } 2}} q_i \vec{v}_i \, \delta(\vec{r}-\vec{r}_i)$)

What is $\vec{B}_1$ from $\vec{J}_1$? Use Biot-Savart law

we rederive the Biot-Savart Law!

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$
$$= -\vec{\nabla}^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \mu_0 \vec{J}$$

use Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$

$$\Rightarrow -\vec{\nabla}^2\vec{A} = \mu_0\vec{J}$$

solution of Poisson's eqn for a localized source

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

now get field

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int d^3r' \, \vec{\nabla}\times\left[\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right]$$

acts on coord $\vec{r}$

use $\vec{\nabla} \times (f\vec{g}) = f(\vec{\nabla}\times\vec{g}) - \vec{g}\times\vec{\nabla} f$

here $\quad \vec{g} = \vec{j}(\vec{r}\,')$ indep of $\vec{r}$

$\qquad f = \dfrac{1}{|\vec{r}-\vec{r}\,'|}$

$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\int d^3r'\ \vec{j}(\vec{r}\,') \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

use $\quad -\vec{\nabla}\left(\dfrac{1}{|\vec{r}-\vec{r}\,'|}\right) = \dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$

Coulomb potential of point charge $\qquad$ field of point charge

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \vec{j}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \qquad \underline{\text{Biot-Savart}}$$

so field $\vec{B}_1$ from current $\vec{j}_1$ is

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r_1\ \vec{j}_1(\vec{r}_1)\times\frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$

plug into expression for $\vec{F}_2$

$$\vec{F}_2 = \int d^3r_2\ \vec{j}_2(\vec{r}_2)\times\vec{B}_1(\vec{r}_2)$$

$$= \frac{\mu_0}{4\pi}\int d^3r_2 \int d^3r_1\ \vec{j}_2(r_2)\times\frac{[\vec{j}_1(\vec{r}_1)\times(\vec{r}_2-\vec{r}_1)]}{|\vec{r}_2-\vec{r}_1|^3}$$

use triple cross product rule

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\Rightarrow \vec{j}_2\times[\vec{j}_1\times(\vec{r}_2-\vec{r}_1)] = \vec{j}_1(\vec{j}_2\cdot(\vec{r}_2-\vec{r}_1)) - (\vec{r}_2-\vec{r}_1)(\vec{j}_2\cdot\vec{j}_1)$$