$$\vec{F}_2 = \frac{\mu_0}{4\pi} \int d^3 r_2 \int d^3 r_1 \; \vec{J}_1(\vec{r}_1) \left[\vec{J}_2(\vec{r}_2) \cdot \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \right]$$

$$- \frac{\mu_0}{4\pi} \int d^3 r_2 \int d^3 r_1 \; \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \; \vec{J}_2(\vec{r}_2) \cdot \vec{J}_1(\vec{r}_1)$$

Consider the first term

acts on $\vec{r}_2$

$$\int d^3 r_2 \; \vec{J}_2(\vec{r}_2) \cdot \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} = - \int d^3 r_2 \; \vec{J}_2(\vec{r}_2) \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right)$$

integrate by parts $\quad \vec{\nabla} \cdot (f \vec{g}) = f \vec{\nabla} \cdot \vec{g} + \vec{g} \cdot \vec{\nabla} f$

$$\Rightarrow \vec{g} \cdot \vec{\nabla} f = \vec{\nabla} \cdot (f \vec{g}) - f \vec{\nabla} \cdot \vec{g}$$

$$= - \int d^3 r_2 \left[\vec{\nabla}_2 \cdot \left(\frac{\vec{J}_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|^3} \right) - \frac{1}{|\vec{r}_2 - \vec{r}_1|} \left(\vec{\nabla}_2 \cdot \vec{J}_2(\vec{r}_2) \right) \right]$$

acts on $\vec{r}_2$ $\qquad$ $\vec{\nabla}_2 \cdot \vec{J}_2(\vec{r}_2) = 0$ since we are in magnetostatic situation

$$= - \oint_S d\vec{a} \cdot \frac{\vec{J}_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|^3}$$

use Gauss' theorem to convert to surface integral

S boundary of system $\to \infty$

surface integral $\to 0$ as $S \to \infty$ since $\vec{J}_2$ is localized

So 1st term vanishes and

$$\vec{F}_2 = - \frac{\mu_0}{4\pi} \int d^3 r_2 \int d^3 r_1 \; \vec{J}_2(\vec{r}_2) \cdot \vec{J}_1(\vec{r}_1) \; \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} \int d^3r_1 \int d^3r_2 \, \vec{j}_2(\vec{r}_2)\cdot\vec{j}_1(\vec{r}_1) \, \frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}$$

for current loops this becomes

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} d\vec{l}_1 \cdot d\vec{l}_2 \, \frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}$$

Note, we can get the force on loop 1 due to the current in loop 2 just by interchanging $1 \leftrightarrow 2$ in above formula. As expected, we then get $\vec{F}_1 = -\vec{F}_2$

Now let $\vec{R}$ be center of mass of loop 2 and $\vec{r}_2{}'$ the relative coordinate. So $\vec{r}_2 = \vec{R} + \vec{r}_2{}'$ for points on loop 2. We can then write

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \, \frac{(R + r_2{}' - \vec{r}_1)}{|\vec{R}+\vec{r}_2{}'-\vec{r}_1|^3}$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \, \vec{\nabla}_R \left(\frac{1}{|\vec{R}+\vec{r}_2{}'-\vec{r}_1|} \right)$$

acts on $\vec{R}$

The work we have to do to move loop 2 in from $\vec{R} = \infty$ to $\vec{R} = R_0$ is

$$\tilde{W} = -\int_{\infty}^{\vec{R}_0} d\vec{R} \cdot \vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \int_{\infty}^{R_0} d\vec{R} \cdot \vec{\nabla}_R \left(\frac{1}{|\vec{R}+\vec{r}_2{}'-\vec{r}_1|} \right)$$

(−) sign as this is the work we must do against the electromagnetic force between the two loops

$$\tilde{W} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \frac{1}{|\vec{R}_0 + \vec{r}_2' - \vec{r}_1|}$$

$$= \frac{-\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|}$$

$$\tilde{W} = -M I_1 I_2$$

add on the work to turn loop currents up to I_1 and I_2 when separated infinitely apart and we get

$$\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 - M I_1 I_2$$

<u>BUT</u> in previous way of computing energy (move loops into position, THEN turn up currents) we found the work to be

$$\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

In method ① the interaction energy of the loops is $+M I_1 I_2$

In method ② it seems that the interaction energy of the loops is $-M I_1 I_2$

How do we reconcile these two answers? The energy stored in the configuration should not depend on the process used to construct the configuration if energy is conserved!

Solution: when we constructed the configuration according to method ② we assumed the currents I_1 and I_2 in the two loops stayed constant as the loop 2 was moved into position with respect to loop 1. But as loop 2 moves, the flux through loop 2 due to current in loop 1 changes $\Rightarrow$ emf induced in loop 2. Similarly, flux through loop 1 changes $\Rightarrow$ emf induced in loop 1.

If we want to keep I_1 and I_2 constant there must be some battery in each loop doing work to counter these induced emfs. The work done by these batteries is

$$\frac{dW_{battery}}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2 \qquad \mathcal{E}_1 = \frac{-d\Phi_1}{dt}$$

$$= I_1 \frac{d\Phi_1}{dt} + I_2 \frac{d\Phi_2}{dt} \qquad \mathcal{E}_2 = \frac{-d\Phi_2}{dt}$$

$$W_{battery} = \int_0^T dt \left(I_1 \frac{d\Phi_1}{dt} + I_2 \frac{d\Phi_2}{dt} \right) = I_1 \Phi_1 + I_2 \Phi_2$$

integrate from $t=0$ when loops infinitely separated to $t=T$ when loops in final position.

As loop moves, I_1 and I_2 stay constant but Φ_1 and Φ_2 change

$$W_{battery} = I_1 \Phi_1 + I_2 \Phi_2$$

Φ_1 is flux through loop 1 due to field from loop 2.

$$= I_1 (M I_2) + I_2 (M I_1)$$

$$= 2 M I_1 I_2$$

Total work is $\tilde{W} + W_{battery} = -M I_1 I_2 + 2 M I_1 I_2 = M I_1 I_2$

Moral: Even in a magnetostatic configuration such as two loops with steady currents, we need to know about dynamics, ie Faraday's law, in order to compute the magnetostatic energy stored in the configuration

Levi-Civita tensor

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{when } ijk \text{ is an even permutation of } 123 \\ -1 & \text{when } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise, in particular if any of the two } ijk \text{ are equal} \end{cases}$$

ijk is an <u>even</u> permutation of 123 if you can get to it by making an <u>even</u> number of pairwise interchanges. An <u>odd</u> permutation requires an <u>odd</u> number of pairwise interchanges.

213 is an ~~even~~ odd permutation $123 \to 213$ one switch $\equiv$ 1 interchange

231 is an ~~odd~~ even permutation $123 \to 213 \to 231$ switch switch $\equiv$ 2 interchanges

~~[illegible]~~

If $\vec{A} = \bar{B} \times \bar{C}$ then

ith component of A is $A_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} B_j C_k$

For example $(1 = x,\ 2 = y,\ 3 = z)$

$$A_1 = \sum_{jk} \varepsilon_{1jk} B_j C_k$$

By properties of ε_{ijk}, the only terms in above sum that are not zero are $(j, k) = (2,3)$ and $(3,2)$

$\varepsilon_{123} = +1, \quad \varepsilon_{132} = -1$

So

$$A_1 = B_2 C_3 - B_3 C_2$$ this is just the x-component of $\bar{B} \times \bar{C}$

you can check that the other components also come out correct

$$A_2 = B_3 C_1 - B_1 C_3 \quad , \quad A_3 = B_1 C_2 - B_2 C_1$$

A useful relation is

$$\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Since $\varepsilon_{ijk} = 0$ unless $i\,j\,k$ are all different the above will be non zero only if the pair (j,k) has the same numbers as the pair (l, m), and $j \neq k$ and $l \neq m$. When $j=l$ and $k=m$, then the above is $(\varepsilon_{ijk})^2 = +1$. When $j=m$ and $k=l$, then the above is $\varepsilon_{ijk}\varepsilon_{ikj} = -1$.

You can check that the right hand side obeys all these properties

Example: $\bar{A} \times (\bar{B} \times \bar{C})$

i^{th} component of the above is

$$\sum_{jklm=1}^{3} \varepsilon_{ijk} A_j \underbrace{\varepsilon_{klm} B_l C_m}_{k^{th} \text{ component of } \bar{B}\times\bar{C}}$$

$$= \sum_{jklm} \varepsilon_{kij} \varepsilon_{klm} A_j B_l C_m = \sum_{jklm} \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right] A_j B_l C_m$$

$\varepsilon_{ijk} = \varepsilon_{kij}$ since 2 pair interchanges take from ijk to kij

$$\sum_{jklm} \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \right] A_j B_l C_m$$

$$= \sum_j A_j B_i C_j - \sum_j A_j B_j C_i$$

$$= B_i (\vec{A} \cdot \vec{C}) - C_i (\vec{A} \cdot \vec{B})$$ as $\vec{A} \cdot \vec{C} = \sum_j A_j C_j$

so

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

BAC − CAB rule!