electromagnetic waves in a vacuum : for $\rho = \vec{j} = 0$

1) $\vec{\nabla}\cdot\vec{E} = 0$ 3) $\vec{\nabla}\cdot\vec{B} = 0$

2) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{B} = \cancel{\mu_0 \vec{j}} + \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\vec{\nabla}\times(2) \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0 \text{ as } \rho=0}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

from (4)

$$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\right)\vec{E} = \Box^2\vec{E} = 0.$$

Similarly $\vec{\nabla}\times(4) \Rightarrow$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{0}) - \vec{\nabla}^2 B = \mu_0\varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

(2)

$$= \mu_0\varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \left(\nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\right)\vec{B} = \Box^2\vec{B} = 0.$$

for any function $f(\vec{r}, t)$,

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$ is "wave equation"

describes waves moving with speed v.

$\Rightarrow$ Maxwell's equ's in vacuum have wave solutions that move with speed $= \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3\times 10^8$ m/s.

This combination of $\varepsilon_0\mu_0$ turns out to be exactly the speed of light! This realization of Maxwell's demonstrated that light was just an electro-magnetic wave!

Solutions to the wave equation $\Box^2 f = 0$.

plane waves: if $g(\phi)$ is any function of single variable ϕ, then

$f(\vec{r},t) \equiv g(\vec{k}\cdot\vec{r} - \omega t)$ solves wave equation, for any vector $\vec{k}$, and for $\omega^2 = v^2 k^2$

proof:

$$\Box^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial g}{\partial \phi} k_x \quad \text{chain rule}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial x}\right) = \frac{\partial g}{\partial \phi}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 g}{\partial \phi^2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{\partial^2 g}{\partial \phi^2} k_x^2$$

similarly $\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial \phi^2} k_y^2$, $\frac{\partial^2 g}{\partial z^2} = \frac{\partial^2 g}{\partial \phi^2} k_z^2$, $\frac{\partial^2 g}{\partial t^2} = \frac{\partial^2 g}{\partial \phi^2} \omega^2$

$$\Box^2 f = \left(k^2 - \frac{\omega^2}{v^2}\right)\frac{\partial^2 g}{\partial \phi^2} = 0 \quad \text{if} \quad \omega^2 = v^2 k^2. \quad \text{for any function } g(\phi)$$

~~ω is frequency of the wave, i.e. ν = ω/2π, T = 1/ν is period~~

$\cancel{f(r, t+T) = f(\vec{k}\cdot\vec{r} - \omega(t+T))}$

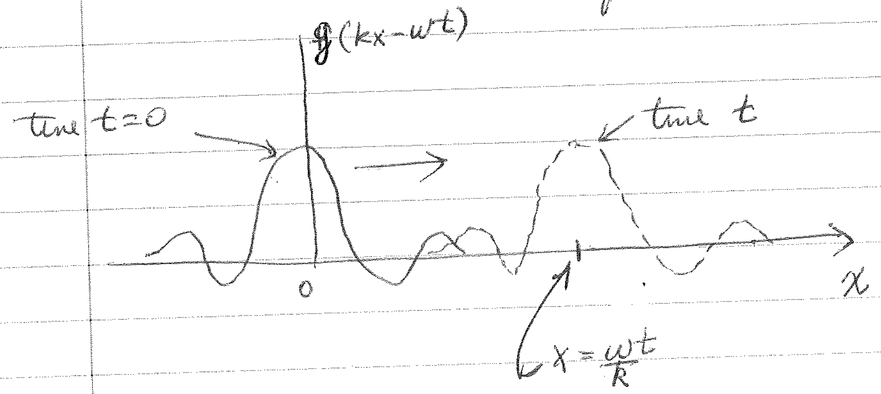
$\cancel{= f(\vec{k}\cdot\vec{r} - \omega t - \omega\frac{2\pi}{\omega})}$

plane wave because $f(\vec{r},t)$ is constant on all planes $\perp$ to $\vec{k}$

i.e. if $\Delta \vec{r}$ is $\perp$ to $\vec{k}$, then

$$f(\vec{r}+\Delta\vec{r}, t) = g(\vec{k}\cdot\vec{r} + \underbrace{\vec{k}\cdot\Delta\vec{r}}_{=0} - \omega t) = g(\vec{k}\cdot\vec{r} - \omega t)$$
$$= f(\vec{r}, t)$$

v is ~~velocit~~ speed of wave, suppose $\vec{k} = k\hat{x}$



curve shifted to right a distance $x = \frac{\omega}{k} t$ in time t.
$\Rightarrow$ curve moves with velocity $\vec{v} = \frac{\omega}{k}\hat{x}$

spherical waves

solution to wave eqn that depends only on radial coord r, ie $f(r,t)$ ← f const on spheres of radius r

$$\left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] f(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

try solution of form $\boxed{f(r,t) = \frac{g(kr-\omega t)}{r}}$

call $\phi = kr - \omega t$

$$r^2\left(\frac{\partial f}{\partial r}\right) = r^2\left(\frac{1}{r}\frac{dg}{d\phi}k - \frac{g}{r^2}\right) = r\frac{dg}{d\phi}k - g$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\left[\frac{dg}{d\phi}k + r\frac{d^2g}{d\phi^2}k^2 - \frac{dg}{d\phi}k\right] = \frac{d^2g}{d\phi^2}\frac{k^2}{r}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{r}\frac{d^2g}{d\phi^2}\omega^2$$

so $\frac{g}{r}(kr-\omega t)$ solves wave eqn provided $\frac{d^2g}{d\phi^2}\frac{k^2}{r} = \frac{1}{v^2}\frac{d^2g}{d\phi^2}\frac{\omega^2}{r}$

ie $\omega^2 = v^2k^2$

Note: if $f_1(\vec{r},t)$ and $f_2(\vec{r},t)$ are solutions to wave eqn, then so is $f_1 + f_2$, as $\Box^2$ is a linear operator

Back to plane waves

A particular solution : sinusoidal wave

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

period of wave is $T = \frac{2\pi}{\omega}$: $f(\vec{r}, t+T) = f(t)$
frequency of wave is $\nu = \omega/2\pi = 1/T$
angular freq is ω

wavelength is $\lambda = \frac{2\pi}{|\vec{k}|}$: $f(\vec{r}+\lambda\hat{k}, t) = f(\vec{r},t)$
wave vector is $\vec{k}$ wavenumber is $k = |\vec{k}|$

amplitude is A
phase constant is δ

phase velocity is $\vec{v} = \frac{\omega}{|\vec{k}|}\hat{k}$ travels in direction $\hat{k}$

ex: if $\vec{k} = k\hat{x}$ then wave travels in $+\hat{x}$ direction
if $\vec{k} = -k\hat{x}$ then $A\cos(-kx - \omega t + \delta)$
travels in $-\hat{x}$ direction

Complex notation $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(x,t) = \mathrm{Re}\left[A e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta)}\right]$$

usually we will not write the "Re" ~~by~~ but leave it implied.

$$f(x,t) = \mathrm{Re}\Big[\underbrace{A e^{i\delta}}_{\text{complex amplitude}} e^{i(\vec{k}\cdot\vec{r} - \omega t)}\Big]$$

Sinusoidal waves particularly important, since we can expand any solution of wave eqn in terms of them
$\Rightarrow$ theory of Fourier Transforms

Fourier Series: Any function $f(x)$ defined on $[-\frac{L}{2}, \frac{L}{2}]$ can be expressed as

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right) \right]$$

where $$A_n = \frac{2}{L} \int_{-\frac{L}{2}}^{L/2} dx\, f(x) \cos\left(\frac{2\pi n x}{L}\right)$$

$$B_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) \sin\left(\frac{2\pi n x}{L}\right)$$

rewrite in terms of complex exponential.

$$\cos\theta = \frac{e^{i\theta} + e^{i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ \left(\frac{A_n}{2} + \frac{B_n}{2i}\right) e^{i\frac{2\pi n x}{L}} + \left(\frac{A_n}{2} - \frac{B_n}{2i}\right) e^{-i\frac{2\pi n x}{L}} \right\}$$

where

$$\frac{A_n}{2} \pm \frac{B_n}{2i} = \frac{A_n}{2} \mp i\frac{B_n}{2} = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, \frac{1}{2} \left\{ \cos\frac{2\pi n x}{L} \mp i \sin\frac{2\pi n x}{L} \right\}$$

$$= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{\mp i \frac{2\pi n x}{L}}$$

$n = 1, 2, \ldots$

Define $f_n \equiv \frac{A_n}{2} + \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi n x}{L}}$

$f_{-n} \equiv \frac{A_n}{2} - \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi(-n)x}{L}}$

$f_0 \equiv \frac{A_0}{2} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)$

$$f(x) = f_0 + \sum_{n=1}^{\infty}\left\{ f_n e^{i\frac{2\pi n x}{L}} + f_{-n} e^{i 2\pi(-n)x/L}\right\}$$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i 2\pi n x/L}, \quad f_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi n x}{L}}}$$

Fourier series in Complex form

Now let $L \to \infty$

Define $k_n = \frac{2\pi n}{L}$, $k_{n+1} - k_n = \Delta k = \frac{2\pi}{L}$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} L\, f_n\, e^{i k_n x}$$

Define $\tilde{f}(k_n) \equiv \frac{L}{2\pi} f_n = \frac{1}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i k_n x}$

$$f(x) = \sum_{n=-\infty}^{\infty} \Delta k\, \tilde{f}(k_n)\, e^{i k_n x}$$

as $L \to \infty$, $\Delta k \to 0$, $\sum \Delta k \to \int dk$

$$\boxed{f(x) = \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{ikx}, \quad \tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}}$$

Fourier transform

$\tilde{f}(k)$ is the Fourier transform of $f(x)$

$$f = f^*$$

If $f(x)$ is a real function, then

$$\tilde{f}(-k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{+ikx} = \int_{-\infty}^{\infty} dx\, f^*(x) \left(e^{-ikx}\right)^*$$

$$= \left(\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}\right)^* = \tilde{f}^*(-k)$$

$$\boxed{\tilde{f}(-k) = \tilde{f}^*(k)}$$

For a function of 3-dim space,

$$f(x,y,z) = \int_{-\infty}^{\infty} dk_x\, dk_y\, dk_z\, \tilde{f}(k_x, k_y, k_z)\, e^{ik_x x} e^{ik_y y} e^{ik_z z}$$

or $$f(\vec{r}) = \int_{-\infty}^{\infty} d^3k\, \tilde{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

where $$\tilde{f}(\vec{k}) = \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3}\, f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}$$

For function of $\vec{r}$ and t

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where
$$\tilde{f}(\vec{k},\omega) = \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, f(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$$

Fourier transforms are of enormous help in solving partial differential equations.