$\Rightarrow R_\perp = R_\parallel = 1$ ~~when material b~~

this confirms that material b is totally reflecting in this region of frequency

When medium b is <u>transparent</u>, ie ε_b is real and $\varepsilon_b > 0$ we have

$$k_{Iz} = \omega\sqrt{\mu_a \varepsilon_a}\cos\theta_I = \frac{\omega}{c} n_a \cos\theta_I$$

$$k_{Tz} = \omega\sqrt{\mu_b \varepsilon_b}\cos\theta_T = \frac{\omega}{c} n_b \cos\theta_T$$

and Snell's Law applies, so $n_a \sin\theta_I = n_b \sin\theta_T \Rightarrow \frac{n_b}{n_a} = \frac{\sin\theta_I}{\sin\theta_T}$

we can now write $R_\perp$ and $R_\parallel$ as functions of θ_I

For simplicity take $\mu_a = \mu_b = \mu_0$

① $$R_\perp = \left(\frac{n_a\cos\theta_I - n_b\cos\theta_T}{n_a\cos\theta_I + n_b\cos\theta_T}\right)^2 = \left(\frac{\cos\theta_I - \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T}\right)^2$$

$$= \left(\frac{\sin\theta_T\cos\theta_I - \sin\theta_I\cos\theta_T}{\sin\theta_T\cos\theta_I + \sin\theta_I\cos\theta_T}\right)^2 = \left(\frac{\sin(\theta_I - \theta_T)}{\sin(\theta_I + \theta_T)}\right)^2$$

for $\theta_I = 0$, ie normal incidence, $\theta_I = \theta_T = 0$

$$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$$

if $n_a = n_b$, no reflection!

(2) $$R_{\parallel} = \left(\frac{\varepsilon_b n_a \cos\theta_I - \varepsilon_a n_b \cos\theta_T}{\varepsilon_b n_a \cos\theta_I + \varepsilon_a n_b \cos\theta_T} \right)^2$$

use $\sqrt{\varepsilon_b \mu_0} = \frac{n_b}{c}$

$\Rightarrow \varepsilon_b = \frac{n_b^2}{c^2 \mu_0} = n_b^2 \varepsilon_0$

$\varepsilon_a = n_a^2 \varepsilon_0$

$$= \left(\frac{n_b \cos\theta_I - n_a \cos\theta_T}{n_b \cos\theta_I + n_a \cos\theta_T} \right)^2$$

$$= \left(\frac{\cos\theta_I - \left(\frac{\sin\theta_T}{\sin\theta_I}\right) \cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_T}{\sin\theta_I}\right) \cos\theta_T} \right)^2 = \left(\frac{\sin\theta_I \cos\theta_I - \sin\theta_T \cos\theta_T}{\sin\theta_I \cos\theta_I + \sin\theta_T \cos\theta_T} \right)^2$$

$$R_{\parallel} = \left(\frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)} \right)^2 \Leftarrow \text{after some algebra}$$

for $\underline{\theta_I = 0} \Rightarrow \theta_T = 0$

$$R_{\parallel} = \left(\frac{\varepsilon_b n_a - \varepsilon_a n_b}{\varepsilon_b n_a + \varepsilon_a n_b} \right)^2 = \left(\frac{n_b - n_a}{n_b + n_a} \right)^2$$

same as for $R_\perp$! But when $\theta_I = 0$ there is no distinction between the two case $\perp$ and $\parallel$, so this is to be expected.

when $\theta_I + \theta_T = \pi/2$, then
$\tan(\theta_I + \theta_T) \to \infty$ and $R_{\parallel} = 0$.

This occurs at an angle of incidence $\theta_I \equiv \theta_B$ "Brewster's angle"

θ_B determined by $\underset{\theta_I}{n_a \sin\theta_B} = n_b \sin\underbrace{\left(\frac{\pi}{2} - \theta_B\right)}_{=\theta_T} = n_b \cos\theta_B$

$$\Rightarrow \boxed{\tan\theta_B = \frac{n_b}{n_a}}$$

For a wave incident at θ_B, the reflected wave will always have $\vec{E}_R \perp$ plane of incidence, no matter what orientation of incoming $\vec{E}_I$, since $R_{\parallel} = 0$. ~~That is only $R_{\perp} \neq 0$, so reflected wave can only have $\vec{E} \perp$ plane of incidence.~~

If incoming wave has component of $\vec{E}_I \parallel$ to plane of incidence, this component gets purely transmitted since $R_{\parallel} = 0$. Only the component of $\vec{E}_I \perp$ to plane of incidence can get reflected, since $R_{\perp} \neq 0$. $\Rightarrow$ reflected wave is polarized with $\vec{E}_R \perp$ to plane of incidence.

Generally, for all θ_I close to θ_B, $R_{\parallel} \ll R_{\perp}$, and the reflected wave is strongly polarized with $\vec{E}_R$ mostly $\perp$ to plane of incidence.

This is therefore one method to create a polarized light wave.

Radiation from moving charges

In Lorentz gauge: $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$

potentials solve $\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \equiv \Box^2 V = -\rho/\varepsilon_0$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t} \equiv \Box^2\vec{A} = -\mu_0\vec{J}$$

if we know potentials, can get fields from

$$\vec{B} = \vec{\nabla}\times\vec{A}, \quad \vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

As in electro and magneto statics, it is easier to solve for V and $\vec{A}$ and then determine $\vec{E}$ and $\vec{B}$, rather than try to solve for $\vec{E}$ and $\vec{B}$ directly.

Recall solutions for statics: $\nabla^2 V = -\rho/\varepsilon_0$, $\nabla^2\vec{A} = -\mu_0\vec{J}$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(r')}{|r-r'|} \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')}{|r-r'|}\,d^3r'$$

Both solutions follow from the fact that

$$\nabla^2\left(\frac{1}{|r-r'|}\right) = -4\pi\,\underbrace{\delta(\vec{r}-\vec{r}')}_{\text{Dirac }\delta\text{-function}}$$

$\frac{1}{|r-r'|}$ is called the "Green's function" for the operator ∇^2

Similarly, if we could find the "Green's function" for the $\Box^2$ operator, ie a function $G(\vec{r}-\vec{r}', t-t')$ that solved

$$\Box^2 G(\vec{r}-\vec{r}', t-t') = -4\pi\, \delta(\vec{r}-\vec{r}')\, \delta(t-t')$$

then the solutions to $\Box^2 V = -\rho/\varepsilon_0$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$, would be

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\, G(\vec{r}-\vec{r}', t-t')\, \rho(\vec{r}', t')$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\, G(\vec{r}-\vec{r}', t-t)\, \vec{J}(r', t')$$

How to find $G(\vec{r}-\vec{r}', t-t')$? Use Fourier transf method

$$G(\vec{r},t) = \int d^3k \int d\omega\, \tilde{G}(k,\omega)\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}$$

$$\delta(\vec{r}) = \int d^3k\, \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3} \qquad \text{from HW}$$

$$\delta(t) = \int d\omega\, \frac{e^{-i\omega t}}{2\pi}$$

substitute into $\Box^2 G(\vec{r},t) = -4\pi\, \delta(\vec{r})\, \delta(t)$

$$\int d^3k \int d\omega\, \tilde{G}(k,\omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t} = -4\pi \int d^3k \int d\omega\, \frac{e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}}{(2\pi)^4}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\int d^3k \int d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \left(\frac{\omega^2}{c^2}-k^2\right)\tilde{G}(k,\omega) = \int d^3k \int d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \frac{(-4\pi)}{(2\pi)^4}$$

equate Fourier coefficients

$$\Rightarrow \left(\frac{\omega^2}{c^2}-k^2\right)\tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4}$$

$$\tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4}\frac{c^2}{(\omega^2-c^2k^2)}$$

$$G(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \tilde{G}(k,\omega)$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} \frac{d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{(2\pi)^4} \frac{(-)4\pi c^2}{(\omega+ck)(\omega-ck)}$$

integrand diverges when $\omega = \pm ck$

Can evaluate using methods of complex contour integration (see complex variables course)

$$G(\vec{r},t) = \begin{cases} 0 & t<0 \\ \frac{c}{r}\delta(r-ct) & t>0 \end{cases} \qquad \text{where } r = |\vec{r}|$$

$$\boxed{G(\vec{r},t) = \begin{cases} 0 & t<0 \\ \frac{1}{r}\delta\left(t-\frac{r}{c}\right) & t>0 \end{cases}} \qquad \text{as } \delta(ax) = \frac{\delta(x)}{a}$$

$G(\vec{r},t)$ has a reasonable form:

1) $G(\vec{r},t) \sim \delta(|\vec{r}|-ct) \Rightarrow$ response travels with speed c
response from source at $\vec{r}'=0$, $t'=0$,
is only felt at ~~time t = r~~ position $\vec{r}$
at time $t = \frac{r}{c}$ later.

2) If take $c \to \infty$, $G(\vec{r},t) \to \frac{\delta(t)}{r}$ response instantaneous
and $\frac{1}{r}$ is Greens function of ∇^2

expected as $\lim_{c\to\infty} \Box^2 = \nabla^2$

Explicit check that $G = \frac{c}{r}\delta(r-ct)$ solves $\Box^2 G = -4\pi\,\delta(\vec{r})\,\delta(t)$

$\nabla^2(ab) = a\nabla^2 b + b\nabla^2 a + 2\vec{\nabla}a\cdot\vec{\nabla}b$

$$\nabla^2\left[\frac{c}{r}\delta(r-ct)\right] = \frac{c}{r}\nabla^2\delta(r-ct) + 2\,\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) + \delta(r-ct)\,\nabla^2\left(\frac{c}{r}\right)$$

use $\nabla^2\delta(r-ct) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\delta(r-ct)\right)$ in spherical coords

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\delta'(r-ct)\right) = \frac{1}{r^2}\left[2r\,\delta'(r-ct) + r^2\delta''(r-ct)\right]$$

$$= \frac{2}{r}\delta'(r-ct) + \delta''(r-ct)$$ here $\delta'(x) = \frac{d\delta(x)}{dx}$ etc.

$\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) = \left(\frac{-c}{r^2}\right)\left(\delta'(r-ct)\right)$ (evaluated in spherical coords)

$\nabla^2\left(\frac{c}{r}\right) = -4\pi\,\delta(\vec{r})\,c$

so $\nabla^2\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{c}{r}\left(\frac{2}{r}\delta'(r-ct) + \delta''(r-ct)\right)$

$$- \frac{2c}{r^2}\,\delta'(r-ct) \quad - 4\pi c\,\delta(\vec{r})\,\delta(r-ct)$$

$$= \frac{c}{r}\,\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta\!\left(t-\frac{r}{c}\right)$$

using $\delta(r-ct) = \frac{\delta(t-\frac{r}{c})}{c}$

$$= \frac{c}{r}\,\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta(t)$$

since $r = 0$ because of $\delta(r)$ term

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{c}{r}(-c)\,\delta'(r-ct)\right]$$

$$= -\frac{1}{r}\frac{\partial}{\partial t}\,\delta'(r-ct) \;=\; \frac{c}{r}\,\delta''(r-ct)$$

so $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{c}{r}\,\delta(r-ct)\right) = \frac{c}{r}\delta''(r-ct) - 4\pi\,\delta(\vec{r})\,\delta(t)$

$$-\frac{c}{r}\,\delta''(r-ct)$$

$$= -4\pi\,\delta(\vec{r})\,\delta(t) \quad \text{as desired}$$

$$V(\vec{r},t) = \int \frac{d^3r'}{4\pi\varepsilon_0}\,\frac{\rho(r',t')}{|\vec{r}-\vec{r}'|}$$

where $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$ "retarded time"
depends on $\vec{r}$ and $\vec{r}'$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\vec{J}\left(\vec{r}',\, t' = t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

general solution to inhomogeneous wave equation