where $\vec{m}(\omega) \equiv \frac{1}{2}\int d^3r' \, (\vec{r}' \times \vec{J}(\vec{r}', \omega))$ is magnetic dipole moment

$$\overleftrightarrow{Q}'_{ij}(\omega) = \int d^3r' \; 3\vec{r}_i{}'\vec{r}_j{}' \rho(\vec{r}';\omega)$$

looks very close to electric quadrapole tensor

$$\overleftrightarrow{Q}_{ij} = \int d^3r' \left(3\vec{r}_i{}'\vec{r}_j{}' - r'^2 \delta_{ij}\right) \rho(\vec{r}', \omega)$$

$$\overleftrightarrow{Q}'_{ij} = \overleftrightarrow{Q}_{ij} + \delta_{ij}\int d^3r' \, r'^2 \rho(\vec{r}';\omega)$$

$$\vec{I}_2 = -\hat{r} \times \vec{m}(\omega) - \frac{i\omega}{6}\hat{r}\cdot\overleftrightarrow{Q}(\omega) - \frac{i\omega}{6}\hat{r}\underbrace{\int d^3r' \, r'^2 \rho(\vec{r}';\omega)}_{\text{call this } C(\omega) \text{ a scalar}}$$

plug back into $\vec{A}(\vec{r}, \omega)$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{-i\omega\vec{p} - \left(\frac{1}{r} - ik\right)\left(\hat{r}\times\vec{m} + \frac{i\omega}{6}\hat{r}\cdot\overleftrightarrow{Q} + \frac{i\omega}{6}\hat{r}\,C\right)\right\}$$

electric dipole contribution — magnetic dipole contribution — electric quadrapole contribution

The last piece which contributes to $\vec{A}$, ie $\frac{i\omega}{6} c\,\hat{r}\,\frac{e^{ikr}}{r}$
<u>is unimportant</u> - it does not effect the $\vec{E}$ or $\vec{B}$ fields
since

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{\nabla}\times\left[f(r)\,\hat{r}\right] = 0$$

similarly, away from sources, where $\vec{J}=0$, Ampere's law gives

$$\mu_0\varepsilon_0 \frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

$$\shortparallel$$

$$-i\omega\mu_0\varepsilon_0\vec{E}(\vec{r},\omega) = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

since last term doesn't contribute to $\vec{B}$, it doesn't contribute to $\vec{E}$. Formally, we could remove it by making a gauge transformation. Less formally, we will just drop it!

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{-i\omega\vec{p} - \overbrace{\left(\frac{1}{r}-ik\right)}^{=-\left(1+\frac{i}{kr}\right)ik}\left(\hat{r}\times\vec{m} + \frac{i\omega}{6}\,\hat{r}\cdot\overleftrightarrow{Q}\right)\right\}$$

lets look at relative strengths of the different terms

far from sources, $\frac{1}{r}$ will be small compared to k.
<u>radiation zone</u>: just consider those terms in $\vec{A}$ that decrease as slowest powers of $\left(\frac{1}{r}\right)^n$. This will be the $\frac{1}{r}$ terms

<u>Approx ①</u> $d \ll r$
<u>Approx ②</u> $d \ll \lambda$
<u>Radiation Zone</u> $\lambda \ll r$ so $kr \gg 1$

Combine: $d \ll \lambda \ll r$ is RZ

lectric dipole term $\quad \vec{p} \simeq q d$ $\quad$ q is typical charge in source
d is size of source region

magnetic dipole term $\quad \vec{m} = \frac{1}{2}\int d^3r \; \vec{r} \times \vec{j}$ $\quad$ $\vec{j} \sim q v$ where v is typical velocity

$$\simeq d j \simeq d v q \qquad v \sim \frac{d}{\tau} \sim d\omega \sim dck$$

$$\simeq q d^2 \omega \sim q c d^2 k$$

electric quadrapole term $\quad \overleftrightarrow{Q} \sim \int d^3r \; \vec{r}\,\vec{r}\,\rho$

$$\sim q d^2$$

so electric dipole contrib to $\vec{A}$ goes as $\quad \omega \vec{p} \sim q \omega d = qc(kd)$
magnetic dipole contrib to $\vec{A}$ goes as $\quad k\vec{m} \sim q\omega k d^2 = qc(kd)^2$
electric quadrapole contrib to $\vec{A}$ goes as $\quad k\omega \overleftrightarrow{Q} \sim q\omega k d^2 = qc(kd)^2$

Since approx ② assumed (kd) was small
(non relativistic approx: $kd \approx v/c$)
we have an expansion for $\vec{A}$ in powers of (kd)

leading term is the electric dipole term.
next order terms are $\left\{ \begin{array}{l} \text{magnetic dipole} \\ \text{electric quadrapole} \end{array} \right\}$ ← these are comparable in strength.

If we kept higher order terms in our expansion, the next terms would be the magnetic quadrapole and electric octopole, both of order $qc(kd)^3$.

Consider now the leading term, the electric dipole term.

$$\vec{A}_{E1} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\vec{p}(\omega)$$

$E1 \equiv$ electric dipole term

lets now compute the $\vec{E}$ and $\vec{B}$ fields in the electric dipole approx

<u>magnetic field</u>

$$\vec{B}_{E1}(\vec{r},\omega) = \vec{\nabla}\times\vec{A}_{E1}(\vec{r},\omega)$$

$$= -\frac{i\omega\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\vec{p}(\omega)\right)$$

use $\vec{\nabla}\times(f\vec{g}) = f\,\vec{\nabla}\times\vec{g} + (\vec{\nabla}f)\times\vec{g}$ where $f = \frac{e^{ikr}}{r}$, $\vec{g} = \vec{p}$

$$= -\frac{i\omega\mu_0}{4\pi}\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)\times\vec{p}$$

use $\vec{\nabla}f(r) = \frac{\partial f}{\partial r}\hat{r}$

$$= -\frac{i\omega\mu_0}{4\pi}\left\{\left(\frac{ik}{r} - \frac{1}{r^2}\right)e^{ikr}\,\hat{r}\times\vec{p}\right\}$$

use $\omega = ck$

$$= \frac{c\mu_0}{4\pi}k^2\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\hat{r}\times\vec{p}$$

$$\boxed{\vec{B}_{E1} = -\frac{c\mu_0}{4\pi}k^2\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\vec{p}\times\hat{r}}$$

ignore this as r gets large when $kr \gg 1$

ie in <u>radiation zone</u> limit:

$$\boxed{\vec{B}_{E1} = -\frac{c\mu_0}{4\pi}k^2\frac{e^{ikr}}{r}\,\vec{p}\times\hat{r}}$$

electric field

from Ampere, $\mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B}$, since $\vec{J}=0$ far from source

$$\Rightarrow \quad \vec{E}_{E1} = \frac{i}{\omega\mu_0\varepsilon_0} \vec{\nabla}\times\vec{B}_{E1} \qquad \text{since } \frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}$$

$$= \frac{-i}{\omega\mu_0\varepsilon_0}\frac{c\mu_0}{4\pi}k^2\,\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\vec{p}\times\hat{r}\right)$$

$\omega = ck$

to evaluate $\vec{\nabla}\times(\;)$, use $\vec{\nabla}\times(f\vec{g}) = f\,\vec{\nabla}\times\vec{g} + \vec{\nabla}f\times\vec{g}$

with $f = \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)$ and $\vec{g} = \vec{p}\times\hat{r}$

$$\vec{\nabla}\times(\;) = \frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\vec{\nabla}\times(\vec{p}\times\hat{r}) + \vec{\nabla}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right)\times(\vec{p}\times\hat{r})$$

evaluate second term: $\vec{\nabla}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right) = \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\right)\hat{r}$ in spherical coords

$$= e^{ikr}\left[ik\left(\frac{1}{r}+\frac{i}{kr^2}\right) - \frac{1}{r^2} - \frac{2i}{kr^3}\right]\hat{r}$$

$$= \frac{e^{ikr}}{r}\left[ik - \frac{2}{r} - \frac{2i}{kr^2}\right]\hat{r}$$

evaluate first term: $\vec{\nabla}\times(\vec{p}\times\hat{r}) = \vec{p}\,(\vec{\nabla}\cdot\hat{r}) - (\vec{p}\cdot\vec{\nabla})\hat{r}$ cover Griffiths

where $\vec{\nabla}\cdot\hat{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2}{r}$ evaluating in spherical coordinates

and $(\vec{p}\cdot\vec{\nabla})\hat{r} = \sum_k p_k \frac{\partial\hat{r}}{\partial r_k}$

unit vector in k direction

where $\frac{\partial\hat{r}}{\partial r_k} = \frac{\partial}{\partial r_k}\left(\frac{\vec{r}}{r}\right) = \vec{r}\left(-\frac{1}{r^2}\frac{\partial r}{\partial r_k}\right) + \frac{\hat{e}_k}{r}$

$= \vec{r}\left(\frac{-1}{r^2}\frac{r_k}{r}\right) + \frac{\hat{e}_k}{r}$ as $\frac{\partial r}{\partial r_k} = \frac{r_k}{r}$

so $\vec{\nabla}\times(\vec{p}\times\hat{r}) = \dfrac{2\vec{p}}{r} - \sum_k p_k\left(-\dfrac{\vec{r}\,r_k}{r^3} + \dfrac{\hat{e}_k}{r}\right)$

$$= \frac{2\vec{p}}{r} + \frac{\vec{r}\;\vec{p}\cdot\vec{r}}{r^3} - \frac{\vec{p}}{r}$$

$$= \frac{\vec{p} + \hat{r}(\vec{p}\cdot\hat{r})}{r} \qquad \text{using } \hat{r} = \frac{\vec{r}}{r}$$

putting all the pieces together

$$\vec{E}_{E1} = \frac{-ik}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\left(1+\frac{i}{kr}\right)\frac{\vec{p}+\hat{r}(\vec{p}\cdot\hat{r})}{r} + \left(ik - \frac{2}{r} - \frac{2i}{kr^2}\right)\underbrace{\hat{r}\times(\vec{p}\times\hat{r})}_{\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})}\right]$$

order by powers of $\frac{1}{r}$

$$= -\frac{ik}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[ik\left(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})\right) + \frac{1}{r}\left(1+\frac{i}{kr}\right)\left(\vec{p}+\hat{r}(\vec{p}\cdot\hat{r})\right) - \frac{2}{r}\left(1+\frac{i}{kr}\right)\left(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})\right)\right]$$

$$= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(\vec{p}+\hat{r}(\vec{p}\cdot\hat{r}) - 2\vec{p} + 2\hat{r}(\vec{p}\cdot\hat{r})\right)\right]$$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(3\hat{r}(\vec{p}\cdot\hat{r})-\vec{p}\right)\right]}$$

<u>radiation zone approx</u> $kr \gg 1$ keep only terms of order $\frac{1}{r}$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\underbrace{\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})}_{\hat{r}\times(\vec{p}\times\hat{r})}\right]}$$

radiation zone limit

$$\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\,\frac{e^{ikr}}{r}\;\hat{r}\times(\vec{p}\times\hat{r})$$

$$\vec{B}_{E1} = -\frac{c\mu_0}{4\pi}\,k^2\,\frac{e^{ikr}}{r}\;\vec{p}\times\hat{r}$$

radiation zone fields in electric dipole approx

$\vec{E}$ and $\vec{B}$ are outwards traveling spherical waves $\sim \frac{e^{ikr}}{r}$

using $\mu_0\varepsilon_0 = 1/c^2$

$$\frac{|\vec{B}_{E1}|}{|\vec{E}_{E1}|} = \frac{c\mu_0}{4\pi}\cdot 4\pi\varepsilon_0 = c\mu_0\varepsilon_0 = \frac{c}{c^2} = \frac{1}{c}$$

just as for plane waves in vacuum

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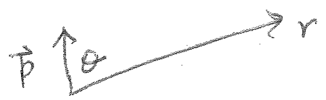
if choose coordinates so that $\vec{p}$ is along $\hat{z}$ axis



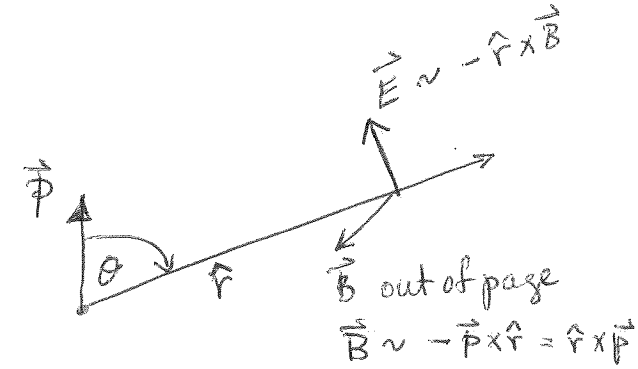
$$\vec{p}\times\hat{r} = \hat{\Phi}\sin\theta\, p$$

$$\hat{r}\times(\vec{p}\times\hat{r}) = p\,\underbrace{\hat{r}\times\hat{\Phi}}_{\sin\theta} = -\hat{\Theta}\sin\theta\, p$$

$$\vec{E}_{E1} = \frac{-k^2 p}{4\pi\varepsilon_0}\,\frac{e^{ikr}}{r}\,\sin\theta\;\hat{\theta}$$

$$\vec{B}_{E1} = -\frac{c\mu_0}{4\pi}\,k^2 p\,\frac{e^{ikr}}{r}\,\sin\theta\;\hat{\phi}$$

$\hat{\theta}$ & $\hat{\phi}$ are spherical coord basis vectors



$\vec{E}$ is in plane containing $\vec{p}$ and $\hat{r}$
$\vec{B}$ is $\perp$ plane containing $\vec{p}$ and $\hat{r}$

$\vec{E}_{E1}$ and $\vec{B}_{E1}$ are orthogonal, as in plane wave,
and both are orthogonal to direction of propagation $\hat{r}$

$\Rightarrow$ oscillating source emits spherical electromagnetic waves.

What is power emitted?

Poynting vector: $\vec{S}_{E1}(\vec{r},t) = \frac{1}{\mu_0} \mathrm{Re}\left[\vec{E}_{E1}(\vec{r},t)\right] \times \mathrm{Re}\left[\vec{B}_{E1}(\vec{r},t)\right]$

$$\mathrm{Re}\left[\vec{E}_{E1}(\vec{r},t)\right] = \mathrm{Re}\left[\frac{k^2}{4\pi\varepsilon_0} \frac{e^{i(kr-\omega t)}}{r} \hat{r}\times(\vec{p}\times\hat{r})\right]$$

$$= \frac{k^2}{4\pi\varepsilon_0} \frac{\cos(kr-\omega t)}{r} \hat{r}\times(\vec{p}\times\hat{r})$$

~~assumg~~ assuming $\vec{p}$ is real

$$\mathrm{Re}\left[\vec{B}_{E1}(\vec{r},t)\right] = -\frac{c\mu_0}{4\pi} k^2 \frac{\cos(kr-\omega t)}{r} (\vec{p}\times\hat{r})$$

$$\vec{S}_{E1} = \frac{-1}{\mu_0} \frac{k^2}{4\pi\varepsilon_0} \frac{c\mu_0 k^2}{4\pi} \frac{\cos^2(kr-\omega t)}{r^2} \left[\hat{r}\times(\vec{p}\times\hat{r})\right]\times\left[\vec{p}\times\hat{r}\right]$$

$$= -\frac{ck^4}{(4\pi)^2\varepsilon_0} \frac{\cos^2(kr-\omega t)}{r^2} \underbrace{\left[\vec{p} - \hat{r}(\hat{r}\cdot\vec{p})\right]\times\left[\vec{p}\times\hat{r}\right]}_{\left(\vec{p}(\vec{p}\cdot\hat{r}) - \hat{r}p^2\right) - (\hat{r}\cdot\vec{p})\left(\vec{p} - \hat{r}(\vec{p}\cdot\hat{r})\right)}$$

$$= \hat{r}\left((\vec{p}\cdot\hat{r}) - p^2\right)$$

$$\vec{S}_{E1} = \frac{ck^4}{(4\pi)^2\varepsilon_0} \frac{\cos^2(kr-\omega t)}{r^2} \left[p^2 - (\vec{p}\cdot\hat{r})^2\right] \hat{r}$$

in radial direction

if define θ as angle between $\vec{p}$ and $\hat{r}$
then $p^2 - (\vec{p}\cdot\hat{r})^2 = p^2(1-\cos^2\theta) = p^2\sin^2\theta$

$$\vec{S}_{E1} = \frac{ck^4 p^2}{(4\pi)^2\varepsilon_0} \frac{\cos^2(kr-\omega t)}{r^2} \sin^2\theta \, \hat{r}$$

points radially outward!

average over one period of oscillation $\langle \cos^2(kr-\omega t)\rangle = \frac{1}{2}$

$$\langle \vec{S}_{E1} \rangle = \frac{ck^4p^2}{2(4\pi)^2\varepsilon_0} \frac{\sin^2\theta}{r^2} \hat{r}$$

note, the $\frac{1}{r^2}$ is important for energy conservation.
If we integrate $\langle \vec{S}_{E1}\rangle$ over the surface of a sphere of radius r, the result is indep of r.

Average $\underbrace{\text{energy flux}}_{=\text{power}}$ flowing through an element of area at spherical angles θ, ϕ is

$$dP_{E1} = \hat{r}\cdot\langle\vec{S}_{E1}\rangle \underbrace{r^2\sin\theta\, d\theta\, d\phi}_{\text{area of surface element } = r^2 d\Omega}$$

$d\Omega = \sin\theta\, d\theta\, d\phi$ differential solid angle

$$= \hat{r}\cdot\langle\vec{S}_{E1}\rangle r^2 d\Omega$$