Abraham - Lorentz Equation

We saw in the hydrogen example, how one can treat radiation-reaction force as a small perturbation of the charges motion when $E_{rad} \ll E_0$ — the electron remained in a circular orbit whose radius decreased slowly compared to period of revolution $1/\omega_0$, as the energy is radiated away.

Now we want a more general equation of motion, including the radiation-reaction force, that will be valid even on time scales $t < \tau$, ie when radiation reaction is not a small perturbation.

Energy conservation

Newton's eqn $m\dot{\vec{v}} = \vec{F}_{ext}$ ← externally applied force

Want to generalize to $m\dot{\vec{v}} = \vec{F}_{ext} + \underbrace{\vec{F}_{rad}}_{\text{radiation-reaction force}}$

We saw that power radiated away by moving charges is $P(t) = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c^3}|\vec{a}(t)|^2$

We therefore would expect that

$$\vec{F}_{rad}\cdot\vec{v} = \left(\vec{F}_{rad}\cdot\frac{d\vec{r}}{dt}\right) = -P(t)$$

work done by $\vec{F}_{rad}$ per unit time = − power charge loses

Not strictly true – $P(t)$ has in it only the power that is radiated away – it does not include the energy flux from the "near zone" or "velocity" fields, which also can instantaneously affect charges motion. But in HW problem we saw that for a periodic motion, the contribution of the "near zone" fields to the energy flux average to zero. So if we integrate over one cycle of the charges motion

$$\int_0^T dt\, \vec{F}_{rad}\cdot\vec{v} = -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\int_0^T dt\left(\frac{d\vec{v}}{dt}\right)^2$$

$$= \frac{-1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\int_0^T dt\, \frac{d\vec{v}}{dt}\cdot\frac{d\vec{v}}{dt}$$

integrate by parts $\quad = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\left[\int_0^T dt\, \frac{d^2\vec{v}}{dt^2}\cdot\vec{v} - \left[\frac{d\vec{v}}{dt}\cdot\vec{v}\right]_0^T\right]$

vanishes, since by definition of cycle period T, $\vec{v}(T)=\vec{v}(0)$, $\dot{\vec{v}}(T)=\dot{\vec{v}}(0)$

$$\Rightarrow \int_0^T dt\left[\vec{F}_{rad} - \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\dot{\vec{a}}\right]\cdot\vec{v} = 0$$

therefore a reasonable guess is

$$\boxed{\vec{F}_{rad} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\dot{\vec{a}}}$$

$= 0$ unless acceleration is changing

Defining time constant $\tau = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2}{c^3 m}$

$$\vec{F}_{rad} = m\tau\dot{\vec{a}}$$

Abraham Lorentz equation:

$$m\vec{a} = \vec{F}_{ext} + \vec{F}_{rad}$$
$$= \vec{F}_{ext} + m\tau\dot{\vec{a}}$$

$$\boxed{m(\vec{a} - \tau\dot{\vec{a}}) = \vec{F}_{ext}}$$

includes time averaged effects of radiation-reation

3rd order in time $\Rightarrow$ to specify a unique solution one needs to know $\vec{r}(t=0)$, $\vec{v}(t=0)$, and $\vec{a}(t=0)$! very different from Newtonian mechanics where all you need to specify is $\vec{r}(t=0)$ and $\vec{v}(t=0)$ to get unique solution.

$(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}})$
$(\vec{v}, \vec{a}, \dot{\vec{a}})$

For example: if $\vec{F}_{ext} = 0$, $\vec{r}(0) = 0$, $\vec{v}(0) = 0$,
Abraham Lorentz equation still gives two possible solutions

$$\vec{a}(t) = \begin{cases} 0 \\ \vec{a}(0)e^{t/\tau} \end{cases} \leftarrow \text{unphysical!}$$

clearly, $\vec{a}(t) = 0$ is the physical solution — a charge at rest, with no force acting on it, should remain at rest — it should not spontaneously accelerate with ~~...~~ exponentially increasing acceleration!

For example, if $\vec{F}_{ext} = 0$, $\vec{r}(0) = 0$, $\vec{v}(0) = 0$,
Solution of Abraham Lorentz eqn is

$$\vec{a}(t) = \vec{a}(0)\, e^{t/\tau} \quad \text{"runaway acceleration"}$$

Clearly this solution is unphysical (a charge a rest, with no force acting on it should remain at rest – not spontaneously accelerate!) unless we also explicitly specify $\vec{a}(0) = 0$.
But even if we reformulate Abraham-Lorentz to remove such runaway solutions, there remain problems.

(see problem 9.30 in text, or Jackson)

Reformulate as follows:

Define
$$\vec{u}(t) \equiv \vec{a}(t)\, e^{-t/\tau}$$
$$\vec{a}(t) = \vec{u}(t)\, e^{t/\tau}$$
$$\mathring{\vec{a}}(t) = \mathring{\vec{u}}(t)\, e^{t/\tau} + \frac{\vec{u}(t)}{\tau}\, e^{t/\tau}$$

$$\Rightarrow \vec{a}(t) - \tau\, \mathring{\vec{a}}(t) = -\tau\, \mathring{\vec{u}}\, e^{t/\tau}$$

$$\text{Abraham Lorentz} \Rightarrow \quad m(\vec{a} - \tau\mathring{\vec{a}}) = -\tau m \mathring{\vec{u}}\, e^{t/\tau} = \vec{F}_{ext}$$

$$\mathring{\vec{u}} = -\frac{1}{m\tau}\, e^{-t/\tau}\, \vec{F}_{ext}$$

integrate
$$\vec{u}(t) - \vec{u}(t_0) = \frac{-1}{m\tau} \int_{t_0}^{t} e^{-t'/\tau}\, \vec{F}_{ext}(t')\, dt'$$

$$\cancel{m(\vec{a}(t) - \vec{a}(t_0)) = \frac{e^{t/\tau}}{\tau} \int_{t}^{t_0} e^{-t'/\tau}\, \vec{F}_{ext}(t')\, dt'}$$

$$\Rightarrow \quad \vec{a}(t)\, e^{-t/\tau} - \vec{a}(t_0)\, e^{-t_0/\tau} = \frac{-1}{m\tau} \int_{t_0}^{t} e^{-t'/\tau}\, \vec{F}_{ext}(t')\, dt'$$

if $\vec{F}_{ext} \equiv 0$ then $\vec{a}(t) = \vec{a}(t_0)\, e^{(t-t_0)/\tau}$

we will get the desired physical solution $\vec{a}(t) = 0$ provided we take $t_0 \to \infty$

$$\vec{u}(t) - \vec{u}(\infty) = -\frac{1}{m\tau} \int_{\infty}^{t} e^{-t'/\tau}\, \vec{F}_{ext}(t')\, dt'$$

$$u(\infty) = \vec{a}(\infty)\, e^{-\infty/\tau} \equiv 0$$

$$\vec{u}(t) = -\frac{1}{m\tau} \int_{\infty}^{t} e^{-t'/\tau}\, \vec{F}_{ext}(t')\, dt'$$

$$m\, \vec{a}(t) = +\frac{1}{\tau} \int_{t}^{\infty} e^{\left(\frac{t-t'}{\tau}\right)}\, \vec{F}_{ext}\, dt'$$

change integration variables $\quad s = \dfrac{t'-t}{\tau} \qquad \tau\, ds = dt'$

$$m\vec{a}(t) = \int_0^{\infty} ds\; e^{-s}\, \vec{F}_{ext}(t + \tau s)$$

in limit $\tau \to 0 \qquad F_{ext}(t+\tau s) \simeq \vec{F}_{ext}(t) + \dot{\vec{F}}_{ext}\, \tau s + \cdots$

$$m\vec{a}(t) = \int_0^{\infty} ds\; e^{-s} \left[\vec{F}_{ext}(t) + \dot{\vec{F}}_{ext}\, \tau s + \cdots \right]$$

$$= \vec{F}_{ext}(t) + \underbrace{\tau\, \dot{\vec{F}}_{ext} \int_0^{\infty} ds\, e^{-s}\, s}_{\to\, 0 \text{ as } \tau \to 0} + o(\tau^2)$$

so recover Newton's eqn in limit $\tau \to 0$

($\tau \to 0 \Rightarrow q \to 0$ ie particle is uncharged $\therefore$ no radiation reaction)

integral-~~differential~~ form of Abraham-Lorentz eqn

$$\boxed{m\vec{a}(t) = \int_0^\infty ds\, e^{-s} \vec{F}_{ext}(t+\tau s)} \leftarrow \text{removes all runaway solutions}$$

gives unique solution with only $\vec{r}(0)$ and $\vec{v}(0)$ specified

But notice that $\vec{a}(t)$ depends not on instantaneous force at time t (as in Newton's eqn), but on the forces ~~tat~~ that will act over time τ into the future! Above eqn violates causality!

Suppose $\vec{F}_{ext} = 0, \; t<0$
$= \vec{F}_0, \; t>0$

$$m\vec{a}(t) = \int_0^\infty ds\, e^{-s} \vec{F}_{ext}(t+\tau s)$$

$$= \int_{s_{min}}^\infty ds\, e^{-s} \vec{F}_0 \quad \text{where} \begin{cases} s_{min} = 0 \text{ for } t>0 \text{, or} \\ t + \tau s_{min} = 0 \Rightarrow \\ s_{min} = -t/\tau \text{ for } t<0 \end{cases}$$

$$= e^{-s_{min}} \vec{F}_0$$

$$m\vec{a}(t) = \begin{Bmatrix} \vec{F}_0 e^{t/\tau} & t<0 \\ \vec{F}_0 & t>0 \end{Bmatrix} = m\dot{\vec{v}}$$

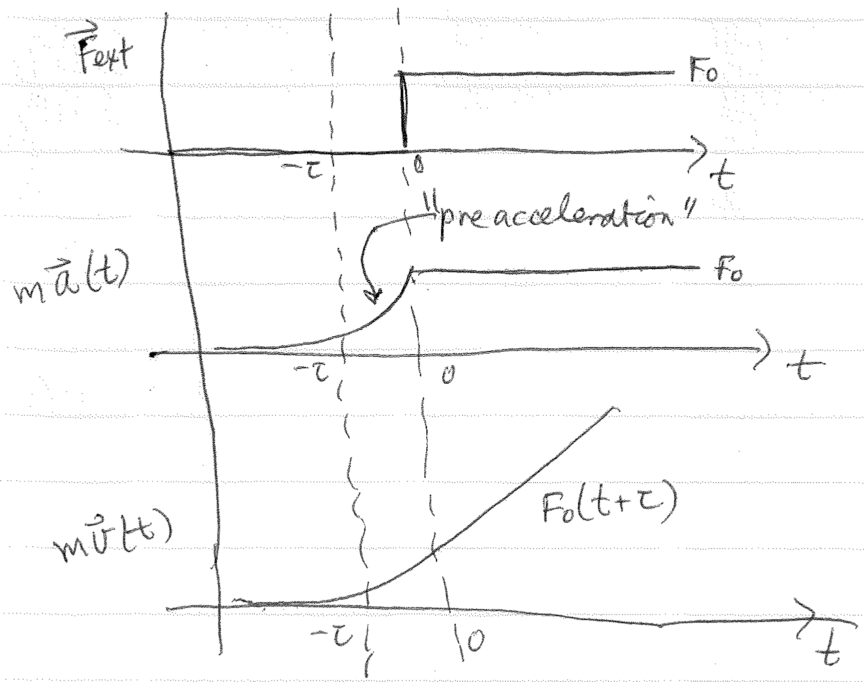
$$\Rightarrow \quad m v(t) = \int_{-\infty}^{t} dt' \, m \dot{\vec{v}}(t') = \int_{-\infty}^{t} dt' \, \vec{F}_0 \, e^{-S_{in}(t')}$$

$$\text{for } t<0 \quad = \int_{-\infty}^{t} dt' \, \vec{F}_0 \, e^{t/\tau} = \vec{F}_0 \tau e^{t/\tau}$$

$$\text{for } t>0 \quad = \int_{-\infty}^{0} dt' \, \vec{F}_0 \, e^{t/\tau} + \int_{0}^{t} dt' \, \vec{F}_0 = \vec{F}_0 \tau + \vec{F}_0 t$$

$$= \vec{F}_0 (t+\tau)$$

$$m\vec{v}(t) = \begin{cases} \vec{F}_0 \tau e^{t/\tau} & t<0 \\ \vec{F}_0 (t+\tau) & t>0 \end{cases}$$



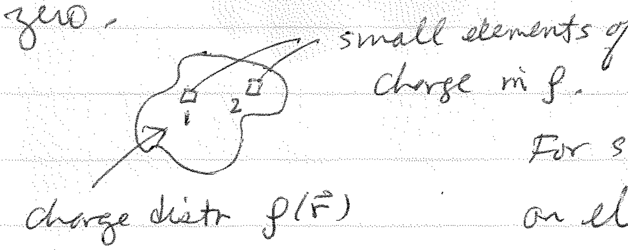
acceleration an velocity start to increase <u>before</u> force is turned on! charge starts to move a time τ <u>before</u> the force acts on it.

For classical problems, this "preacceleration" is unimportant as it only occurs for the extremely small time τ. However this violation of causality is a conceptual theoretical problem with classical E+M + classical mechanics, that has never been resolved. Quantum mechanics also doesnt resolve this problem!

Origin of the Radiation-Reaction Force

An exact arguement which leads to Abraham-Lorentz equation, & explains physical origin of radiation-reaction force is as follows:

Radiation-reaction is the force of a charges $\vec{E}$ + $\vec{B}$ fields, acting back upon the charge itself. These forces are unimportant in statics, as they always balance out to zero.



For static ρ, force of element 1 on element 2, is alway $(-)$ force of element 2 on element 1.

$\Rightarrow$ Net force acting on ρ, due to $\vec{E}$ fields produced by ρ, is always zero!

This is no longer true in dynamics.

The self force on a charge and current distribution is

Lorentz Force $\Rightarrow$ $$\vec{F}_{self} = \int d^3r \left\{ \rho(\vec{r},t)\,\vec{E}_s(\vec{r},t) + \vec{J}(\vec{r},t) \times \vec{B}_s(\vec{r},t) \right\}$$

where $\vec{E}_s$ and $\vec{B}_s$ are the "self" electric and magnetic fields produced by ρ and $\vec{J}$.

1) assume $\rho(\vec{r})$ is spherically symmetric

2) Assume ρ is <u>rigid</u> – doesnt distort as it moves $\Rightarrow \vec{J}(\vec{r},t) = \vec{v}(t)\,\rho(\vec{r},t)$ $\Big\} \Rightarrow$ non relativistic approx

3) assume non relativistic motion

4) Assume charge is instantaneously at rest at time t_0 ie. $\vec{v}(t_0)=0$ (But $\vec{a}(t_0)\neq 0$ in general) (could always transform to a frame of reference in which this is so)

at time t_0:

$$\Rightarrow \quad F_{self} = \int d^3r\, \rho(\vec{r},t_0)\, \vec{E}_s(\vec{r},t_0) \qquad \vec{J}(\vec{r},t_0)=0 \text{ as } \vec{v}(t_0)=0$$

substitute in above to compute $\vec{F}_{self}$

$$\begin{cases} \vec{E}_s(\vec{r},t_0) = -\vec{\nabla} V_s(\vec{r},t_0) - \dfrac{\partial \vec{A}}{\partial t}(\vec{r},t_0) \\[2ex] V_s(\vec{r},t_0) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3r' \,\dfrac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|} \\[2ex] \vec{A}_s(\vec{r},t_0) = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3r' \,\dfrac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|} \end{cases} \qquad t' = t_0 - \frac{|\vec{r}-\vec{r}'|}{c} \qquad \vec{J}(t')\neq 0 \text{ in general}$$

~~Assume particle is small size a_0, moving slowly so that time it takes for fields to travel accross length of particle $\sim \frac{a_0}{c}$ is much less than the ~~distance~~ time the particle takes to move a distance $d \approx a_0$, ie $v \ll c$.~~

~~Then can expand function of t' about t_0.~~

$$\vec{F}_{self}(t_0) = \int d^3r \int d^3r'\, \rho(\vec{r},t_0) \left\{ \frac{-\vec{\nabla}}{4\pi\varepsilon_0}\left(\frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\right) - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\left(\frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\right) \right\}$$

$$\vec{F}_{self} = \int d^3r \int d^3r' \, \frac{\rho(\vec{r},t_0)}{4\pi\varepsilon_0} \left\{ -\vec{\nabla}\left(\frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}(t')\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\right)\right\}$$

If charge is moving non-relativistically $v \ll c$,
After many expansions + much algebra (see Jackson)
one finds

$$\vec{F}_{self} = \underbrace{-\frac{4}{3}\frac{U}{c^2}\dot{\vec{v}}(t_0)}_{\text{inertial term}} + \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{3c^3}\ddot{\vec{v}}(t_0)}_{\text{Abraham Lorentz term}} + \underbrace{(\;)\,a_0}_{\text{nonlinear terms in time derivatives of } \vec{v}}$$

where $U \equiv \cancel{\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int} \; \frac{1}{2}\int d^3r \int d^3r' \, \frac{\rho(\vec{r})\rho(\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|}$

is electrostatic self energy of the charge

a_0 is radius of charge

For a point charge, $a_0 \to 0$, and all terms but first two vanish. Equ of motion becomes

$$m\vec{a} = \vec{F}_{self} + \vec{F}_{ext} = -\frac{4}{3}\frac{U}{c^2}\vec{a} + \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{3c^3}\dot{\vec{a}} + \vec{F}_{ext}$$

or $\left(m + \frac{4}{3}\frac{U}{c^2}\right)\vec{a} = \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{c^3}\dot{\vec{a}} + \vec{F}_{ext}$

two effects: (1) get radiation reaction force $\vec{F}_{rad} = m\tau\dot{\vec{a}}$
as we argued earlier

(2) "renormalize" inertial mass of the charge

$$m_q = m + \frac{4}{3}\frac{U}{c^2} = m + m_{em}$$

Define $\tau = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{3c^3 m_q}$

$$\Rightarrow \quad m_q \vec{a} = m_q \tau \dot{\vec{a}} + \vec{F}_{ext}$$

$$\Rightarrow \quad m_q (\vec{a} - \tau \dot{\vec{a}}) = \vec{F}_{ext}$$

Abraham-Lorentz Equation - only expermental mass m_q appears - now way to detect m or m_{em} separately

m_q = "observed" or "expermental" mass of charge q

$m_{em} = \frac{4}{3}\frac{U}{c^2}$ is the inertial mass associated

with moving the electric and magnetic fields that surround the charge (see HW)

m_{em} is the electromagnetic mass of the ~~part~~ charge

problems with m_{em}

(i) $m_{em} = \frac{4}{3}\frac{U}{c^2}$ but from relativity, would expect to find $m_{em} = \frac{U}{c^2}$. Why the $\frac{4}{3}$?

When assumed $\rho(\vec{r},t)$ was rigid in calculation of $\vec{F}_{self}$, we neglected the mechanical forces needed to keep this rigidity, ie the mechanical forces needed to prevent the small elements of charge that make up the particle from repelling away from each other. These mechanical forces are known as the "Poincaré stresses", and if one includes them, one can make $m_{em} = U/c^2$ consistant with relativity (read in Jackson Chpt 17)

(2) as $a_0 \to 0$ (needed to get rid of the high order terms in eqn of motion), we find

$U \to \infty$ so $m_{em} \to \infty$

$U \sim \frac{1}{4\pi\epsilon_0}\frac{q^2}{a_0}$

But experimentally observed mass of the charge is $(m + m_{em})$ is finite!

$\Rightarrow$ mechanical mass $m \to -\infty$?

If we assumed a_0 was finite, to keep U, finite, and we assumed that <u>all</u> the mass of the ~~electron~~ charge was electromagnetic in origin (ie $m=0$), then

$$a_0 \sim \frac{q^2}{4\pi\epsilon_0 mc^2} = 2.8\times 10^{-13}\ \text{cm for the electron}$$

"classical electron radius"

But it is well established that electron is a <u>point</u> particle at least down to lengths 10^{-15} cm!

Note, classical electron radius

$$a_0 \sim \tau c \qquad \text{where } \tau = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{mc^3}$$

length scale on which radiation-reaction effects are no longer a small perturbation.

<u>Conclusion</u>: Classical mechanics + classical electromagnetism work just fine, provided one stays in the regime where radiation-reaction effects remain only a small perturbation on the motion of charges. If one tries to extend the theory down to length and time scales smaller than this (ie $t<\tau$ or $l<c\tau$) fundamental problems arise having to do with ~~what~~ the ~~is meant by~~ definition of a "point" charge. These problems have <u>never</u> been satisfactorily resolved (Nobel prize is awaiting!)