Inertial frames of reference: Set of frames of reference which move at constant velocity with respect to each other

Special Relativity

1) Speed of light is constant in all inertial frames of reference
2) Physical laws must look the same in all inertial frames of reference - there is no experiment that can determine the "absolute" velocity of any inertial frame

$\Rightarrow$ If a flash of light goes off at the origin of some coord system, the outgoing wave fronts look spherical in all inertial frames.

Equation of wave front is $\quad r^2 - c^2t^2 = 0$

$\Rightarrow$ (x,y,z,t) coords in one inertial frame K,
(x',y',z',t') coords in another inertial frame K' that moves with velocity $\vec{v} = v\hat{x}$ with respect to K.

What is the transformation that relates coords in K' to coords in K

$y = y', \; z = z'$ (origins of K and K' coincide when $t = t' = 0$)

$$\Rightarrow \quad c^2t^2 - x^2 = c^2t'^2 - x'^2$$

$$\Rightarrow \quad \frac{(ct+x)}{(ct'+x')}\frac{(ct-x)}{(ct'-x')} = 1$$

Expect transformation to be *linear*

$$\Rightarrow \quad ct' + x' = (ct+x)\, f$$
$$ct' - x' = (ct-x)\, f^{-1}$$

for some constant f. Write $f = e^{-y}$ $\quad y$ is rapidity

Solve for ct' and x' in terms of ct and x

$$ct' = ct\left(\frac{e^y + e^{-y}}{2}\right) - x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$x' = -ct\left(\frac{e^y - e^{-y}}{2}\right) + x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$ct' = ct\cosh y - x\sinh y$$
$$x' = -ct\sinh y + x\cosh y$$

<u>meaning of parameter y</u>

the origin of K (at $x=0$) has trajectory $x' = -vt'$ in K'

$$\Rightarrow \frac{x'}{t'} = -v$$

from transformation above, with $x=0$, we get

$$\frac{x'}{ct'} = \frac{-ct\sinh y}{ct\cosh y} = -\tanh y$$

so $\frac{v}{c} = \tanh y$

$$\Rightarrow \cosh y = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\sinh y = \left(\frac{v}{c}\right)\gamma$$

<u>Lorentz Transformation</u>
$$\begin{cases} ct' = \gamma ct - \gamma\left(\frac{v}{c}\right)x \\ x' = -\gamma\left(\frac{v}{c}\right)ct + \gamma x \end{cases}$$

Inverse Lorentz transform obtained by taking
$v \to -v$ in above

$$\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right)x' \\ x = \gamma\left(\frac{v}{c}\right)ct' + \gamma x' \end{cases}$$

Time Dilation

Consider a clock located at the origin in frame K' that moves with velocity $v\hat{x}$ as seen from "Lab" frame K.

The clock in frame K' ticks at $t_1' = 0$ and $t_2' = T_0$

Time between ticks in frame K' is thus T_0.

What is time between ticks in frame K?

Since clock is at origin of K', then position of 1st. and 2nd ticks is at $x_1' = x_2' = 0$

In frame K, the observer sees tick 1 at

$$ct_1 = \gamma c t_1' + \gamma\left(\frac{v}{c}\right)x_1' = 0 + 0 = 0$$

and tick 2 at

$$ct_2 = \gamma c t_2' + \gamma\left(\frac{v}{c}\right)x_2' = \gamma c T_0 + 0 = \gamma c T_0$$

So $t_1 = 0$, $t_2 = \gamma T_0$

So time between ticks as seen by K is $\Delta t = \gamma T_0 > T_0$

So it looks to K as if K's clock has slowed down

<u>Proper Time</u> - time between two events as measured in the frame of reference in which those two events occur at the same position

FitzGerald Contraction

Consider frame K' moving with $v\hat{x}$ as seen by K. A ruler, at rest in K', has it's ends located at $x_1' = 0$, $x_2' = L_0$. What is the length of the ruler as seen by K?

At $t=0$ in frame K, the observer measures the positions of the two ends of the ruler and finds

$$x_1' = 0 = -\gamma\left(\frac{v}{c}\right)ct_1 + \gamma x_1 = 0 + \gamma x_1$$

$$\Rightarrow x_1 = 0$$

and

$$x_2' = L_0 = -\gamma\left(\frac{v}{c}\right)ct_2 + \gamma x_2 = 0 + \gamma x_2$$

$$\Rightarrow x_2 = \frac{L_0}{\gamma}$$

So length of ruler in K is $x_2 - x_1 = \frac{L_0}{\gamma} < L_0$

It appears to K as if the ruler has ~~the~~ contracted.

Proper Length - distance between two events as measured in the frame in which the two events happen at the same time

Note: K's measurement of left end occurs at time

$$ct_1' = \gamma c t_1 - \gamma\left(\frac{v}{c}\right) x_1 = 0 \quad \Rightarrow t_1' = 0$$

K's measurement of right end occurs at time

$$ct_2' = \gamma c t_2 - \gamma\left(\frac{v}{c}\right) x_2 = 0 - \gamma\left(\frac{v}{c}\right)\frac{L_0}{\gamma} = -\frac{v}{c} L_0$$

$$t_2' = -\frac{v}{c^2} L_0$$

So K''s interpretation of K's measurement is that K first measures the position of the right end of the ruler, and only a time $\frac{v}{c^2}L_0$ later measures the location of the left end.

So K' sees K measure a length

$$L_0 - \underbrace{\frac{v^2}{c^2} L_0}_{\text{distance ruler travels between K's two measurements}}$$

$$= L_0\left(1 - \frac{v^2}{c^2}\right) = \frac{L_0}{\gamma^2}$$

So K's two measurements, which are simultaneous to K, do not occur simultaneously to K'.

Events that are simultaneous in one frame of reference are not simultaneous in another frame of reference

So K' sees K measure a length that is according to K' a length equal to $\frac{L_0}{\gamma^2}$.

But K' also sees that K is measuring with a ~~a ruler~~ length scale that is ~~to~~ FitzGerald contracted by a factor $1/\gamma$. So the length $\frac{L_0}{\gamma^2}$ seen by K' looks like the length

$\left(\frac{L_0}{\gamma^2}\right)\frac{1}{(1/\gamma)}$ when K' sees K measure it with K's contracted rulers. Thus K' will agree that K thinks the ruler is $\frac{L_0}{\gamma^2}\gamma = \frac{L_0}{\gamma}$ long.

K thinks the moving ruler has contracted

K' thinks K is both (i) not measuring the ends of the ruler at the same time, and (ii) measuring the length of K''s ruler with K's contracted ruler.

So they can both ~~agree~~ agree on the outcome of what happens, but they ascribe different physical processes to what is happening.

proper time

two events (x_1, t_1) seen in K

(x_2, t_2)

Transform to frame K' in which they are at same position $x_1' = x_2'$. The time $t_2' - t_1'$ in that frame K' is the <u>proper time</u> between the events

$$ct_1' = \gamma c t_1 - \gamma\left(\frac{v}{c}\right)x_1$$

$$ct_2' = \gamma c t_2 - \gamma\left(\frac{v}{c}\right)x_2$$

$$x_1' = -\gamma\left(\frac{v}{c}\right)ct_1 + \gamma x_1$$

$$x_2' = -\gamma\left(\frac{v}{c}\right)ct_2 + \gamma x_2$$

$$x_1' = x_2' \Rightarrow \gamma(x_2 - x_1) - \gamma\left(\frac{v}{c}\right)c\,(t_2 - t_1) = 0$$

$$\Rightarrow \frac{x_2 - x_1}{t_2 - t_1} = v$$

So frame K' travels with $v\hat{x}$ with respect to K.

clearly can have such at K' only if $v < c$.

proper time

The time difference between the events in K' is

$$t_2' - t_1' = \gamma t_2 - \gamma \frac{v}{c^2} x_2 - \gamma t_1 + \gamma \frac{v}{c^2} x_1$$

$$= \gamma\left(t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)\right)$$

$$= \gamma\left(t_2 - t_1 - \frac{v^2}{c^2}\cdot(t_2 - t_1)\right)$$

$$= (t_2 - t_1)\gamma\left(1 - v^2/c^2\right) = (t_2 - t_1)\,\gamma/\gamma^2$$

$$\boxed{\tau \equiv t_2' - t_1' = \frac{t_2 - t_1}{\gamma}}$$

Proper length

two events (x_1, t_1) (x_2, t_2) seen in K
transform to K' in which the occur at same time
$t_1' = t_2'$. The distance $x_2' - x_1'$ in that frame K'
is the <u>proper length</u> between the two events

$$x_1' = -\gamma\left(\frac{v}{c}\right)ct_1 + \gamma x_1$$
$$x_2' = -\gamma\left(\frac{v}{c}\right)ct_2 + \gamma x_2$$

$$ct_1' = \gamma ct_1 - \gamma\left(\frac{v}{c}\right)x_1$$
$$ct_2' = \gamma ct_2 - \gamma\left(\frac{v}{c}\right)x_2$$

$$t_1' = t_2' \Rightarrow \gamma c(t_2 - t_1) - \gamma\left(\frac{v}{c}\right)(x_2 - x_1) = 0$$

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{c^2}{v}$$

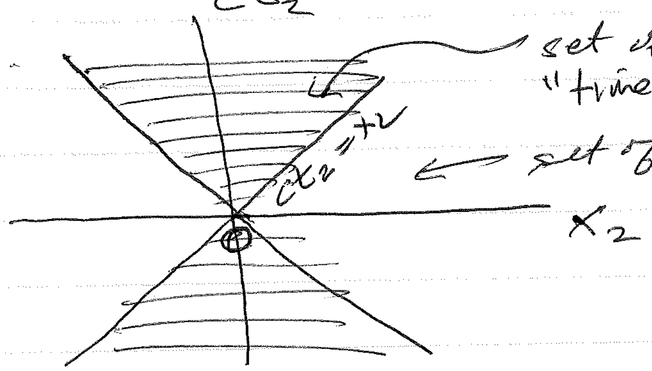
or $v = \dfrac{c^2(t_2 - t_1)}{(x_2 - x_1)}$ such a frame K' can exist only if $v < c$ or $\dfrac{x_2 - x_1}{t_2 - t_1} > c$

Then the proper length is

$$\ell \equiv x_2' - x_1' = \gamma(x_2 - x_1) - \gamma\left(\frac{v}{c}\right)c(t_2 - t_1)$$
$$= \gamma(x_2 - x_1) - \gamma\left(\frac{v}{c}\right)c\,\frac{v}{c^2}(x_2 - x_1)$$
$$= (x_2 - x_1)\gamma\left(1 - \frac{v^2}{c^2}\right) = (x_2 - x_1)\gamma/\gamma^2$$

$$\boxed{\ell = \frac{x_2 - x_1}{\gamma}}$$

Consider two events, one of which occurs at $(x_1 = 0, t_1 = 0)$ and the other at (x_2, t_2)



set of points such that $\frac{x_2}{t_2} < c$ "time-like" region

set of points such that $\frac{x_2}{t_2} > c$ "space-like" region

The time-like region $\frac{x_2}{t_2} < c$ consists of all points such that there is a frame in which x_2 occurs at the same position as x_1 and we can therefore define the proper time between the two events.
Time-like region is such that a pulse of light emitted at origin at $t_1 = 0$ will arrive at position x_2 at a time <u>earlier</u> than t_2.

The space-like region $\frac{x_2}{t_2} > c$ consists of all points such that there is a frame in which t_2 occurs at the same time as t_1 and we can therefore define the proper length between the two events
Space-like region is such that a pulse of light emitted at origin at $t_1 = 0$ will arrive at position x_2 at a time <u>later</u> than t_2

The light cone $\frac{x_2}{t_2} = c$ separates the time-like from the space-like regions. The pt at origin can effect only events in its future time-like region. It is effected only by events in its past time-like region

~~Inverse transform obtained by taking $v \to -v$ in above~~

$$\cancel{\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right)x' \\ x = \gamma\left(\frac{v}{c}\right)ct' + \gamma x' \end{cases}}$$

4 - vectors

4-position: $x_\mu = (x_1, x_2, x_3, ict)$ $\qquad x_4 \equiv ict$

summation convention $x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2t^2$ Lorentz invariant scalar
- sum over repeated indicies

– has same value in all interial frames

Lorentz transf is

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 + i\left(\tfrac{v}{c}\right)x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\left(\tfrac{v}{c}\right)x_1\right) \end{aligned}\right\}$$

linear transf, can be represented by a matrix

or $\quad x_\mu' = a_{\mu\nu}(L)\, x_\nu$

↑ matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $\quad x_\mu = a_{\mu\nu}(L^{-1})\, x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $\quad a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose $\Rightarrow$ "orthogonal"