$F_{\mu\nu}$ transforms under a Lorentz transformation just like a tensor (ie not like a vector)

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$

use $A'_\nu = a_{\nu\lambda} A_\lambda$, $\frac{\partial}{\partial x'_\mu} = a_{\mu\sigma} \frac{\partial}{\partial x_\sigma}$ since A_μ and $\frac{\partial}{\partial x_\sigma}$ are both 4-vectors

$$F'_{\mu\nu} = a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma} - a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$= a_{\mu\sigma} a_{\nu\lambda} \left(\frac{\partial A_\lambda}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\lambda} \right)$$

$$\boxed{F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}}$$ ← transformation law for a 2nd rank tensor

In terms of matrix multiplication, and writing for the transpose of a matrix $a_{\nu\lambda} = a^t_{\lambda\nu}$, the above can b written as

$$F'_{\mu\nu} = a_{\mu\sigma} F_{\sigma\lambda} a^t_{\lambda\nu}$$

The above has the form of the product of three matrices

If we write out the above transformation law component by component we get the following transformation law for the $\vec{E}$ and $\vec{B}$ fields.
For a transformation from K to K', where K' moves with velocity $v\hat{x}$ as seen from K,

$$E'_1 = E_1 \qquad B'_1 = B_1$$
$$E'_2 = \gamma(E_2 - vB_3) \qquad B'_2 = \gamma\left(B_2 + \frac{v}{c^2} E_3\right)$$
$$E'_3 = \gamma(E_3 + vB_2) \qquad B'_3 = \gamma\left(B_3 - \frac{v}{c^2} E_2\right)$$

where $(1,2,3) = (x,y,z)$

The transformation law for an n^{th} rank tensor is

$$T'_{\mu_1, \mu_2, \ldots \mu_n} = a_{\mu_1 \nu_1} a_{\mu_2 a_2} \cdots a_{\mu_n \nu_n} T_{\nu_1, \nu_2, \ldots, \nu_n}$$

Inhomogeneous Maxwell's Equations

Using the field strength tensor $F_{\mu\nu}$ we can write the inhomogeneous Maxwell's equations (ie the ones involving the sources ρ and $\vec{J}$) as follows:

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu}$$

$F_{\mu\nu}$ is a 4-tensor 2nd rank

$\frac{\partial}{\partial x_\nu}$ is a 4-vector

$\Rightarrow \frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector

Proof that $\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector. Using the transformation laws of $F_{\mu\nu}$ and $\frac{\partial}{\partial x_\nu}$ we get

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} a_{\nu\sigma} a_{\nu\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau}$$

write $\sum_\nu a_{\nu\sigma} a_{\nu\tau} = \sum_\nu a^t_{\sigma\nu} a_{\nu\tau}$

but since a is orthogonal $a^t = a^{-1}$ and $\sum_\nu a^t_{\sigma\nu} a_{\nu\tau} = \delta_{\sigma\tau}$

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} \delta_{\sigma\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau} = a_{\mu\lambda} \frac{F_{\lambda\sigma}}{\partial x_\sigma}$$

so transforms like a 4-vector

back to $\boxed{\dfrac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu}$

To see this is so, substitute in definition of $F_{\mu\nu}$ in terms of 4-potential A_μ

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

1st term = 0 by Lorentz gauge condition. So

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = -\Box^2 A_\mu = \mu_0 j_\mu$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu \Rightarrow \begin{cases} \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} & \text{spatial components} \\ \vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \rho/\varepsilon_0 & \text{temporal component} \end{cases}$$

We still need to have a Lorentz covariant way to write the homogeneous Maxwell Equations.

$\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$

<u>Homogeneous Maxwell Equations</u>

Construct the 3rd rank co-variant tensor

$$\boxed{\tilde{G}_{\mu\nu\lambda} \equiv \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}}$$

transforms as $\tilde{G}_{\mu\nu\lambda} = a_{\mu\alpha} a_{\nu\beta} a_{\lambda\gamma} \tilde{G}_{\alpha\beta\gamma}$

$\tilde{G}_{\mu\nu\lambda}$ has in principle $4^3 = 64$ components

But can show that $\tilde{G}$ is antisymmetric in exchange of any two indicies

$$\tilde{G}_{\nu\mu\lambda} = \frac{\partial F_{\nu\mu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\lambda}}{\partial x_\nu} \qquad \text{but since } F_{\mu\nu} = -F_{\nu\mu}$$

$$= -\frac{\partial F_{\mu\nu}}{\partial x_\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x_\mu} - \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = -G_{\mu\nu\lambda}$$

$\Rightarrow$ $\tilde{G}_{\nu\mu\lambda} = 0$ if any two indicies are equal

$\Rightarrow$ there are only $\underline{\underline{4}}$ independent components of $G_{\mu\nu\lambda}$

these are

$$\tilde{G}_{123}, \ \tilde{G}_{124}, \ \tilde{G}_{134}, \ \tilde{G}_{234}$$

all other components are just equal to $\pm$ one of these according to permutation of indicies.

The 4 homogeneous Maxwell equations can be written as

$$\boxed{\tilde{G}_{\mu\nu\lambda} = 0}$$

To see that above is true, substitute in for $F_{\mu\nu}$ in terms of potential A_μ in defintion of $\tilde{G}$

$$\tilde{G}_{\mu\nu\lambda} = \underbrace{\frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} \underbrace{- \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda}}_{\text{cancell}} \underbrace{- \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu}}_{\text{cancele}} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}}_{\text{cancell}}$$

Also, one has

$$\tilde{G}_{123} = \frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{23}}{\partial x_1} = \frac{\partial B_3}{\partial x_3} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_1}{\partial x_1} = 0$$

$$\tilde{G}_{123} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\tilde{G}_{412} = \frac{\partial F_{41}}{\partial x_2} + \frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_4} = \frac{i\,\partial E_1}{c\,\partial x_2} + \frac{-i\,\partial E_2}{c\,\partial x_1} + \frac{\partial B_3}{ic\,\partial t}$$

$$= \frac{i}{c}\left[\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} - \frac{\partial B_3}{\partial t}\right] = -\frac{i}{c}\left[(\vec{\nabla}\times\vec{E})_3 + \frac{\partial B_3}{\partial t}\right] = 0$$

this is the z-component of Faraday's law $\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$\tilde{G}_{413} = 0$ and $\tilde{G}_{423} = 0$ give x and y components of Faraday's law.

An alternative way to write the homogeneous Maxwell's Equations

Note: we can get the homogeneous Maxwell's equations from the inhomogeneous equations by making the substitutions

$$\vec{j} \to 0, \quad \rho \to 0, \quad \frac{\vec{E}}{c} \to \vec{B}, \quad \vec{B} \to -\frac{\vec{E}}{c}$$

so we define the <u>dual field strength tensor</u>

$$G_{\mu\nu} = \begin{pmatrix} 0 & -E_3/c & E_2/c & -iB_1 \\ E_3/c & 0 & -E_1/c & -iB_2 \\ -E_2/c & E_1/c & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$$

or equivalently if

$$\varepsilon_{\mu\nu\sigma\lambda} = \begin{cases} +1 & \text{if } \mu\nu\sigma\lambda \text{ is an even permutation of } 1234 \\ -1 & \text{if } \mu\nu\sigma\lambda \text{ is an odd permutation of } 1234 \\ 0 & \text{otherwise, ie any two indices equal} \end{cases}$$

generalization of the Levi-Civita symbol

then $$\boxed{G_{\mu\nu} = \frac{1}{2i}\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}}$$

psuedo-tensor gives wrong sign under parity transform.

Then $$\boxed{\frac{\partial G_{\mu\nu}}{\partial x_\nu} = 0}$$

gives the homogeneous Maxwell's equations

From $F_{\mu\nu}$ and $G_{\mu\nu}$ we can construct the following Lorentz invariant scalars

$$\frac{1}{2}F_{\mu\nu}F_{\mu\nu} = B^2 - \frac{E^2}{c^2}$$

$$-\frac{1}{4}F_{\mu\nu}G_{\mu\nu} = \frac{\vec{B}\cdot\vec{E}}{c}$$

these have the same value in any inertial frame of reference!

$\Rightarrow$ i) If $\vec{E} \perp \vec{B}$ and $|\vec{B}| = \frac{|\vec{E}|}{c}$ in one frame of reference, then it is so in all frames of reference.
($\vec{E}\cdot\vec{B} = 0$ and $|\vec{B}|^2 - \frac{|\vec{E}|^2}{c^2} = 0$)
This property is satisfied by EM waves in the vacuum

2) If in one frame $\vec{E}\cdot\vec{B} = 0$ and $\frac{E^2}{c^2} > B^2$, then there exists a frame in which $\vec{B}' = 0$. If in one frame $\vec{E}\cdot\vec{B} = 0$ and $B^2 > E^2/c^2$, then there exists a frame in which $\vec{E}' = 0$.

Relativistic Kinematics

4-momentum of a particle $\quad p_\mu = m\,\dot{x}_\mu = m u_\mu = (m\gamma\vec{v},\ i m c\gamma)$

m is mass of particle as measured in the frame in which the particle is instantaneous at rest. m = "rest mass"

p_μ is a 4-vector since m is a scalar and u_μ is a 4-vector

$$p_\mu^2 = m^2 u_\mu^2 = -m^2c^2 \qquad \text{since } u_\mu^2 = -c^2$$

4-force $\quad K_\mu = (\vec{K}, i K_0)$ also called "Minkowski force"

We guess that the relativistic generalization of Newton's 2nd law of motion is

$$m\frac{d^2x_\mu}{ds^2} = K_\mu \quad \text{or} \quad m\frac{du_\mu}{ds} = K_\mu$$

$$\text{or} \quad \frac{dp_\mu}{ds} = K_\mu \qquad (p_\mu = m u_\mu = m\dot{x}_\mu)$$

Now since $p_\mu^2 = -m^2c^2$ is a constant, we have

$$0 = \frac{d}{ds}(p_\mu^2) = 2p_\mu\frac{dp_\mu}{ds} = 2p_\mu K_\mu$$

$$\Rightarrow p_\mu K_\mu = 0$$

$$p_\mu K_\mu = m\gamma\vec{v}\cdot\vec{K} - mc\gamma K_0 = 0$$

$$\text{so} \quad \boxed{K_0 = \frac{\vec{v}}{c}\cdot\vec{K}}$$

time component of 4-force is determined by the spacial components $\vec{K}$

Define the usual 3-force by

$$\frac{d\vec{p}}{dt} = \vec{F}$$ (we identify the Newtonian momentum $\vec{p}$ with the spatial components of p_μ)

$$\frac{d\vec{p}}{ds} = \vec{K}$$ spatial part of relativistic Newton's law

$$\frac{d\vec{p}}{ds} = \gamma \frac{d\vec{p}}{dt} = \gamma \vec{F}$$ since $ds = dt/\gamma$

$$\Rightarrow \boxed{\vec{K} = \gamma \vec{F}}$$ relation between spatial part of 4-force and the usual 3-force $\vec{F}$

$$\Rightarrow K_0 = \frac{\vec{v}}{c} \cdot \vec{K} = \gamma \frac{\vec{v}}{c} \cdot \vec{F}$$

Consider now the 4-th component of Newton's equation

$$\frac{dp_4}{ds} = m \frac{du_4}{ds} = m \frac{d}{ds}(ic\gamma) = i K_0 = i \gamma \frac{\vec{v}}{c} \cdot \vec{F}$$

$$\Rightarrow \frac{d}{ds}(m\gamma c^2) = \gamma \vec{v} \cdot \vec{F}$$

$$d(m\gamma c^2) = \gamma \vec{v} \cdot \vec{F}\, ds = \gamma \vec{v} \cdot \vec{F} \frac{dt}{\gamma}$$

$$= \vec{v} \cdot \vec{F}\, dt = d\vec{r} \cdot \vec{F}$$

$\Rightarrow$

Work-energy theorem: $d(m\gamma c^2) = d\vec{r} \cdot \vec{F}$

work done on particle

$\Rightarrow$ change in kinetic energy of particle

relativistic kinetic energy $\boxed{E = m\gamma c^2}$

$p_4 = im\gamma c = iE/c$

$p_\mu = (\vec{p}, \frac{iE}{c})$ momentum-energy 4-vector

$\vec{p} = m\gamma\vec{v}$

$E = m\gamma c^2$

For particles moving at non-relativistic speeds $v \ll c$

$$E = m\gamma c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \simeq \frac{mc^2}{1 - \frac{v^2}{2c^2}} \simeq mc^2\left(1 + \frac{v^2}{2c^2}\right)$$

$$\simeq mc^2 + \tfrac{1}{2}mv^2$$

rest mass energy ↑ ↑ non-relativistic kinetic energy

$\frac{dp_\mu}{ds} = K_\mu$ is therefore both the relativistic analogue of Newton's 2nd law, but also the law of conservation of energy (ie the work-energy theorem)

Conservation of momentum and energy

1. why is relativistic momentum ~~[illegible]~~ $\vec{p} = m\gamma\vec{v}$ and not just $m\vec{v}$ as in non-relativistic case?

Because we want momentum to be conserved in <u>all</u> frames of reference, $\vec{p}$ must be the spatial part of a 4-vector. We see this as follows.

Suppose momentum was $m\vec{v}$. For a collection of particles, conservation of momentum would mean

$$(*) \qquad \sum_i m_i \vec{v}_i(t_1) = \sum_i m_i \vec{v}_i(t_2)$$

for any times t_1 and t_2

If $(*)$ holds in one frame of reference K, and we now transform to another frame of reference K' moving with velocity $\vec{w}$ wrt K, we would find that in K', $(*)$ is no longer satisfied

ie $$\sum_i m_i \vec{v}_i{}'(t_1) \neq \sum_i m_i \vec{v}_i{}'(t_2)$$

$\vec{v}_i{}'$ related to $\vec{v}_i$ and $\vec{w}$ via relativistic law for addition of velocities

see Griffiths ~~chapt 10.2~~ 2 example 12.6

However, for the 4-momentum, if

$$P_\mu^{tot}(t_1) = \sum_i P_{\mu i}(t_1) = \sum_i P_{\mu i}(t_2) = P_\mu^{tot}(t_2)$$

in frame K, then $P_\mu'^{tot}(t_1) = P_\mu'^{tot}(t_2)$ in any other frame K', since $P_\mu^{tot}(t_1)$ and $P_\mu^{tot}(t_2)$ both transform the same way under Lorentz transf.

$$p_\mu^{tot}(t_1) = p_\mu^{tot}(t_2)$$

space components $\Rightarrow$ momentum conservation holds in all frames
time component $\Rightarrow$ energy conservation holds in all frames

(2) Why did we write Newton's eqn as $\frac{d\vec{p}}{dt} = \vec{F}$, with $\vec{p} = m\gamma\vec{v}$,

instead of $m\frac{d\vec{v}}{dt} = \vec{F}$ (as if used non-relativistic momentum)

If use $m\frac{d\vec{v}}{dt} = \vec{F}$, then $m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\vec{F}$

$$\frac{1}{2}m\,d(v^2) = dt\,\vec{v}\cdot\vec{F} = d\vec{r}\cdot\vec{F} = dW$$

$$\Rightarrow \frac{1}{2}m\int d(v^2) = \int dW$$

$\frac{1}{2}mv^2 = W$ get non-relativistic kinetic energy

in this formulation, energy W is not the time component of any 4-vector. Therefore if energy was conserved in one frame K, it need not be conserved in another frame K'!

Only when we take $\frac{d\vec{p}}{dt} = \vec{F}$ with $\vec{p} = m\gamma\vec{v}$

do we get $\int \vec{F}\cdot d\vec{r} = m\gamma c^2 = p_0 c$ · time component of a 4-vector

$\Rightarrow$ energy conservation holds in all reference frames