Lorentz force in relativistic form

$$\frac{dp_\mu}{ds} = K_\mu$$

what is the K_μ that represents the Lorentz force? And how can we write it in a Lorentz covariant way?

K_μ should depend on the fields $F_{\mu\nu}$ and on the particle's trajectory x_μ

as $\vec{v} \to 0$ $\vec{K} = q\vec{E}$ (since magnetic force $\to 0$ as $\vec{v} \to 0$)

K_μ can't depend directly on x_μ as the force should be independent of where one puts the origin of the coordinates. So K_μ can depend only on derivatives $\dot{x}_\mu$, $\ddot{x}_\mu$, etc.

As $\vec{v} \to 0$, $\vec{K}$ does not depend on the acceleration, so $\vec{K}$ does not depend on $\ddot{x}_\mu$ or higher derivatives.

So K_μ depends only on $F_{\mu\nu}$ and $\dot{x}_\mu$

We need to form a 4-vector out of $F_{\mu\nu}$ and $\dot{x}_\mu$ that is <u>linear</u> in the fields $F_{\mu\nu}$ and proportional to the charge q. (since we want superposition to hold)

The only possibility is

$$K_\mu = q\, f(\dot{x}_\mu^2)\, F_{\mu\nu}\, \dot{x}_\nu$$

where $f(\mathring{x}_\mu^2)$ is some function of $\mathring{x}_\mu^2$.
But $\mathring{x}_\mu^2 = -c^2$ is a <u>constant</u>, so $f(\mathring{x}_\mu^2)$ is a constant.
That constant, $f(\mathring{x}_\mu^2) = 1$, is determined by the requirement
that $\vec{K} = q\vec{E}$ as $\vec{v} \to 0$.

So we have $$\boxed{K_\mu = q F_{\mu\nu} \mathring{x}_\nu}$$

Let's check what this gives for the ordinary 3-force
$\vec{F} = \frac{1}{\gamma}\vec{K}$

ith component $$F_i = \frac{1}{\gamma} K_i = \frac{q}{\gamma}\left(\sum_{j=1}^{3} F_{ij}\mathring{x}_j + F_{i4}\mathring{x}_4\right)$$

sub in for F_{ij} in terms of $\vec{A}$ $$= \frac{q}{\gamma}\left(\sum_{j=1}^{3}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\mathring{x}_j - \frac{iE_i}{c}(ic\gamma)\right)$$
use $\mathring{x}_4 = ic\gamma$

since $F_{i4} = \frac{-iE_i}{c}$

Now $$\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \varepsilon_{ijk} B_k$$

<u>proof:</u> $$\varepsilon_{ijk} B_k = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial A_m}{\partial x_l}$$ (using ε_{ijk} notation to take $\vec{\nabla}\times\vec{A}$)

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial A_m}{\partial x_l}$$

$$= \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$$

use $\mathring{x}_j = \gamma v_j$

So $$F_i = \frac{q}{\gamma}\sum_{j=1}^{3}\varepsilon_{ijk}B_k\gamma v_j + \frac{q}{\gamma}E_i\gamma$$

$$= q\sum_{j=1}^{3}\varepsilon_{ijk}B_k v_j + qE_i$$

$$= qE_i + q(\vec{v}\times\vec{B})_i$$

so $\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} = \frac{1}{\gamma}\vec{K}$

The Lorentz force has the same form in all inertial frames.
No relativistic modification is needed

Relativistic Larmor's formula

non relativistic result was

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}a^2$$ total power radiated by particle with acceleration $\vec{a}$ assuming $v \ll c$

Now consider a particle moving with any speed v.
Consider the inertial frame of reference in which that particle is instantaneously at rest. Call this frame $\mathring{K}$. The velocity in this frame is thus $\mathring{\vec{v}} = 0$, and the charge is at the origin ~~$\vec{r}=0$~~ $\mathring{\vec{r}} = 0$.

The power radiated, as seen in the frame $\mathring{K}$, is then exactly

$$\mathring{P} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}\mathring{a}^2$$ where $\mathring{a}$ is the acceleration in frame $\mathring{K}$.

This result is exact because as $v/c \to 0$ all terms higher than the electric dipole term will vanish.

What we need to do is to find the way to Lorentz transform the result $\mathring{P}$ and find its value in any other frame of reference, in which the particle is moving with any velocity $\vec{v}$.

Consider the momentum-energy 4-vector describing the total momentum and total energy of the electromagnetic fields ~~of the charge~~ radiated by the charge.

in frame $\overset{\circ}{K}$ we can write this as

$$\left(\overset{\circ}{\vec{P}}_{EM},\ \frac{i\overset{\circ}{\mathcal{E}}}{c}\right)$$

Now $\overset{\circ}{\vec{P}}_{EM} = \int d^3\overset{\circ}{r}\ \varepsilon_0 \overset{\circ}{\vec{E}} \times \overset{\circ}{\vec{B}}$.

But since the radiated fields are in the radial direction $\overset{\circ}{\hat{r}}$, when we integrate over all space we find $\overset{\circ}{\vec{P}}_{EM} = 0$.

Alternatively you have from homework, for a charge moving with small velocity $\vec{v}$, $\vec{P}_{EM} = \frac{4}{3}\frac{U}{c^2}\vec{v}$
So when $\vec{v} \to 0$, $\vec{P}_{EM} \to 0$.

So in frame $\overset{\circ}{K}$ the momentum energy 4-vector is

$$\left(0,\ \frac{i\overset{\circ}{\mathcal{E}}}{c}\right)$$

In a new frame of reference K that moves with velocity $-\vec{v}$ with respect to $\overset{\circ}{K}$ (in frame K, the charge is moving with velocity $\vec{v}$) the energy in frame K is obtained by the transformation law for 4-vectors

$$\frac{i\mathcal{E}}{c} = \gamma\left(\frac{i\overset{\circ}{\mathcal{E}}}{c} + \frac{iv}{c}\overset{\circ}{P}_{EM\,\parallel}\right)$$

where $\overset{\circ}{P}_{EM\,\parallel}$ is component of $\overset{\circ}{\vec{P}}_{EM}$ in direction of $\vec{v}$. But $\overset{\circ}{\vec{P}}_{EM} = 0$

So $\quad i\frac{\mathcal{E}}{c} = \gamma i \frac{\mathring{\mathcal{E}}}{c} \quad \Rightarrow \quad \mathcal{E} = \gamma \mathring{\mathcal{E}}$

similarly, if we take the origins of K and $\mathring{K}$ to coincide at the time when we are measuring the radiated power, then time transforms as

$$t = \gamma \mathring{t} + \frac{v}{c^2}\gamma \mathring{x}_1$$

where $\mathring{x}_1$ is position of charge in direction of $\vec{v}$

But charge is at origin in $\mathring{K}$ so $\mathring{x}_1 = 0$

So $\quad t = \gamma \mathring{t} \quad \leftarrow$ (since charge is not moving in $\mathring{K}$, $d\mathring{t}$ is really the proper time ds, so this is the familiar $\frac{dt}{\gamma} = ds$)

The Power radiated in frame K is then

$$\mathcal{P} = \frac{d\mathcal{E}}{dt} = \frac{\gamma\, d\mathring{\mathcal{E}}}{\gamma\, d\mathring{t}}$$

transforming $\mathcal{E} = \gamma \mathring{\mathcal{E}}$, $t = \gamma \mathring{t}$

$$= \frac{d\mathring{\mathcal{E}}}{d\mathring{t}} = \mathring{\mathcal{P}}$$

So the total radiated power is a Lorentz invariant scalar!

$$\boxed{\mathcal{P} = \mathring{\mathcal{P}} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3} q \frac{\mathring{a}^2}{c^3}}$$

where $\mathring{a}$ is acceleration of charge in its rest frame

We would like to rewrite $\mathcal{P}$ in a way that makes no explicit reference to the frame $\mathring{K}$.
ie we want to write $\mathring{a}^2$ in terms of a Lorentz invariant scalar that may be evaluated in any frame K.

Consider the 4-acceleration

$$\alpha_\mu = \frac{d u_\mu}{ds} = \gamma \frac{d}{dt} u_\mu \qquad \text{since } ds = dt/\gamma$$

use $u_\mu = (\gamma \vec{v}, i\gamma c)$

$$\vec{\alpha} = \gamma \frac{d}{dt}(\gamma \vec{v}) = \gamma \vec{a} + \vec{v}\frac{d\gamma}{dt}$$

$$\alpha_4 = \gamma i c \frac{d\gamma}{dt}$$

we need $\displaystyle \frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right) = \frac{+\frac{\vec{v}}{c^2}\cdot\frac{d\vec{v}}{dt}}{(1-v^2/c^2)^{3/2}}$

$$= +\frac{\vec{v}\cdot\vec{a}}{c^2}\gamma^3$$

so

$$\vec{\alpha} = \gamma^2 \vec{a} + \gamma^4 \left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v}$$

$$\alpha_4 = \gamma^4 i \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)$$

$$\alpha_\mu = \gamma^4 \left(\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v} + \frac{\vec{a}}{\gamma^2},\ i\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)\right)$$

in frame $\overset{\circ}{K}$, $\overset{\circ}{\vec{v}} = 0$ and $\overset{\circ}{\gamma} = 1$, so

$$\overset{\circ}{\alpha}_\mu = (\overset{\circ}{\vec{a}}, 0) \quad \text{and so} \quad \overset{\circ}{a}{}^2 = \alpha_\mu^2 \quad \text{Lorentz invariant scalar}$$

So now we can write the relativistic Larmor formula

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}\overset{\circ}{a}{}^2 = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}\alpha_\mu^2 \quad \text{in any frame } K$$

In a general frame K,

$$\alpha_\mu^2 = |\vec{\alpha}|^2 + \alpha_4^2$$

$$= \gamma^8 \left[\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)^2 v^2 + \frac{a^2}{\gamma^4} + 2\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\frac{(\vec{v}\cdot\vec{a})}{\gamma^2} - \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

$$= \gamma^8 \left[-\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \left(1 - \frac{v^2}{c^2}\right) + 2\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \frac{1}{\gamma^2} + \frac{a^2}{\gamma^4} \right]$$

$$= \gamma^8 \left[\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \left(\frac{2}{\gamma^2} - \frac{1}{\gamma^2}\right) + \frac{a^2}{\gamma^4} \right]$$

$$= \gamma^8 \left[\frac{a^2}{\gamma^4} + \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \frac{1}{\gamma^2} \right]$$

$$\alpha_\mu^2 = \gamma^4 \left[a^2 + \gamma^2 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

Note: as $v \to 0$, $\gamma \to 1$ and we get $\alpha_\mu^2 = a^2$ as we must.

So power radiated is

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 \left[a^2 + \gamma^2 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

Examples:

① For a charge accelerating in linear motion (such as in a linear particle accelerator such as SLAC)

$\vec{v}\cdot\vec{a} = va$ since $\vec{v}$ and $\vec{a}$ are colinear

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 \left[a^2 + \gamma^2 \frac{v^2 a^2}{c^2} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 a^2 \left[1 + \gamma^2 \frac{v^2}{c^2} \right]$$

$$1+\gamma^2\frac{v^2}{c^2} = 1+\frac{v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2} = \gamma^2$$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}a^2\gamma^6$$

relativistic result increased by factor γ^6 compared to non-relativistic result

② For a charge accelerating in circular motion (such as in a synchrotron)

$\vec{v}\cdot\vec{a} = 0$ since $\vec{v}\perp\vec{a}$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}a^2\gamma^4$$

relativistic result increased by factor γ^4 compared to non-relativistic result