So the velocity $\vec{u}$ of the charges flowing down the wire does <u>not</u> contribute to the emf around the loop.

But $\vec{u}$ is important for something else!

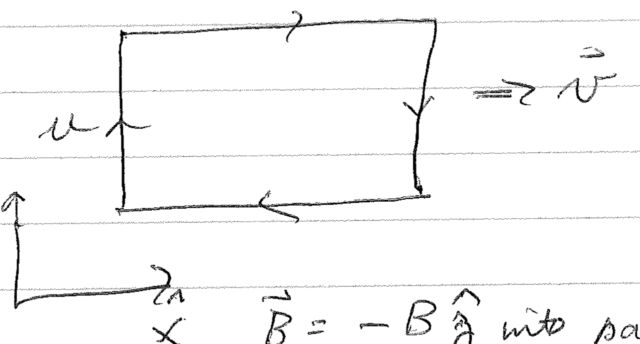
What is the source of the work that drives the current around the loop?

It is not $\vec{B}$ since magnetic fields can do no work!

$$\vec{F}_L = q\vec{v}\times\vec{B} \text{ is } \perp \vec{v} \quad \text{so} \quad \vec{v}\cdot\vec{F}_L = 0$$

$\uparrow$ work per time done by $\vec{F}_L$ on particle

<u>Answer</u>: It is the person pulling the loop that is doing the work



$\vec{B} = -B\hat{z}$ into page

As we computed above, the motion of the charges in the left vertical segment of the loop gives rise to a Lorentz force

$$\vec{F}_L = q\vec{u}\times\vec{B} = q(u\hat{y})\times(-B\hat{z}) = -quB\hat{x}$$

~~The trip down this left segment of the loop takes~~

So the person pulling the loop must exert a force equal and opposite to this, per charge, to pull the loop and keep it moving with constant $\vec{v}$.

$F_{pull} = quB\hat{x}$ ~~Work done by puller, per charge for one trip of charge around loop is~~

Work is only done as the charge moves down left segment, since in other ~~seg~~ horizontal segments $\vec{F}_L$ is in $\hat{y}$ direction and does not oppose force of puller.

Time for charge to flow down left segment is

$$\Delta t = \frac{h}{u}$$

During Δt the loop moves $\Delta s = \Delta t\, v$

so work done by puller to move the charge around the loop is

$$quB\,\Delta s = quB\,\Delta t\, v = quB\left(\frac{h}{u}\right)v$$

$$= q v B h$$

Work per charge ~~[illegible]~~ is

$$W = vBh = \mathcal{E} = -\frac{d\Phi}{dt}$$

So puller does just the right amount of mechanical work to provide the emf driving the current!

~~the trip down the left segment takes a time $\Delta t = \frac{h}{u}$.~~
~~In this time the distance travelled in y direction is $v\Delta t = h\frac{v}{u}$~~
~~ie: $d\vec{l} = h\hat{z} + h\frac{v}{u}\hat{y}$~~

~~$$\int uB\hat{y}\cdot d\vec{l} = uBh\frac{v}{u} = Bhv = \mathcal{E}.$$~~

~~So work done by puller = emf around loop~~

$$\mathcal{E} = -\frac{d\Phi}{dt}$$ is true in <u>general</u> - need not be spatially uniform B, ~~need not be square loop~~ loop may have any shape, or even may change shape as time varies. see proof in text. pg 307-308

<u>Faraday's Law</u>

In previous example, suppose loop is fixed, but magnet moves to left with velocity v. By special relativity, we expect same result as before, ie an emf is induced in loop

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Now what force creates the emf? Since loop (& the charges in it) are <u>stationary</u>, they experience <u>no</u> Lorentz force. Faraday $\Rightarrow$ charges feel an electric force!

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$$

changing B induces an E in loop.

note: since there is no battery, or anything else in the loop driving the current, $\oint \vec{E}\cdot d\vec{l}$ goes completely around loop. But this is inconsistent with electrostatic $\vec{\nabla}\times\vec{E}=0$.
$\Rightarrow$ $\vec{\nabla}\times\vec{E}\neq 0$ in general, when $\vec{B}$ varies in time.

$$\mathcal{E} = -\frac{d\Phi}{dt} \Rightarrow \oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$\parallel$ Stokes

$$\int (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ Faraday's Law.

When $\frac{\partial \vec{B}}{\partial t}=0$, ie statics, we regain $\vec{\nabla}\times\vec{E}=0$.

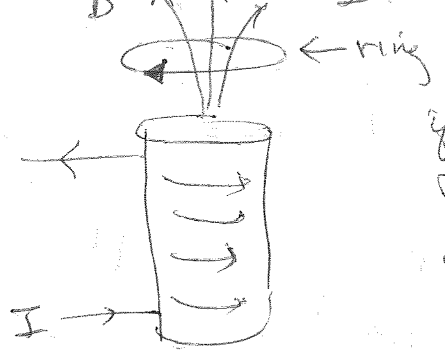
Faraday's law holds true no matter what cause B to change in time, ie same result if magnet moves, or strength of stationary electromagnet is changed by increasing current in solenoid.

Note: If looked at in magnet's rest frame, where loop is moving, the force that moves the charges is magnetic
If looked at in loop's rest frame, where magnet is moving, the force that moves the charges is electric.

$\Rightarrow$ electric forces + magnetic forces depend on the reference frame from which they are viewed
(this is obvious for $\vec{F}_{mag} = q\vec{v}\times\vec{B}$ as relativity $\Rightarrow$ there is no absolute way to measure $\vec{v}$)

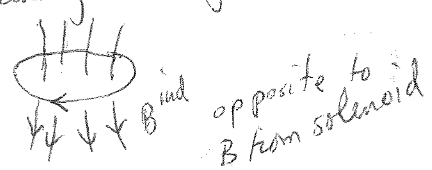
⇒ electric and magnetic fields are part of same physical phenomena. This was one of motivations for Einstein in developing special relativity.

<u>Lenz law</u> $\frac{\partial \Phi}{\partial t}$ induces an emf $\mathcal{E}$ that drives a current in the direction such that the magnetic fields produced by this current, tend to oppose the change in Φ.



if increase current in solenoid, B through ring increases. $\mathcal{E}$ drives current as shown. Bind created by this induced current has negative flux through ring, ie tends to oppose increasing flux from solenoid. Opposite circulating currents in ring + solenoid will repell + ring will jump up!

B field produced by current circulating in ring is



Note: <u>If</u> we have situation where charge density $\rho = 0$, then equ's for $\vec{E}$ are

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

same form as magnetostatics ! $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

we can solve for $\vec{E}$ using methods from magnetostatics!

⇒ $\cancel{\vec{E}(\vec{r}) = -\frac{1}{4\pi}\int d^3r' \frac{\partial \vec{B}(\vec{r}')}{\partial t} \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}$ ~~is solution~~

$\cancel{= -\frac{\partial}{\partial t}\left[\frac{1}{4\pi}\int d^3r' \vec{B}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]}$

Quick review of magnetostatics

$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{B}=\vec{\nabla}\times\vec{A}$ $\quad\vec{A}$ is magnetic vector potential

$\vec{\nabla}\times\vec{B}=\mu_0\vec{j} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A}+\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \mu_0\vec{j}$

choose to work with vector potential $\vec{A}$ that satisfies $\vec{\nabla}\cdot\vec{A}=0$.
we can always do this! see Griffiths pgs 243-244.
using an $\vec{A}$ that satisfies $\vec{\nabla}\cdot\vec{A}=0$ is called
working in the "Coulomb gauge".

$\Rightarrow \quad -\nabla^2\vec{A} = \mu_0\vec{j}$ $\quad$ Poisson's Eqn

looks just like electrostatics $-\nabla^2 V = \rho/\varepsilon_0$

For a <u>localized</u> charge distribution (ie $\rho\to 0$ as $r\to\infty$) in a system that is infinite (ie no boundary walls), we know the solution for the electrostatic potential is given by superposition and Coulomb's law

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

so apply the same form of the solution to the Poisson equation for $\vec{A}$ in magnetostatics

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

applies for a <u>localized</u> current distribution $\vec{j}$ in an infinite system

To get the magnetic field we now use $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \left[\frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right]$$

derivatives act on coordinate $\vec{r}$, ~~not~~ $\vec{r}'$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left[\vec{\nabla} \times \left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) \right]$$

term in the [ ] has the form $\vec{\nabla} \times (\vec{J} f(\vec{r}))$ where $\vec{J}$ is a constant vector, and $f(\vec{r})$ is a scalar function of position. From the front cover of Griffiths we then have

$$\vec{\nabla} \times (f \vec{J}) = f \underbrace{(\vec{\nabla} \times \vec{J})}_{= 0 \text{ when } \vec{J} \text{ is a constant}} - \vec{J} \times (\vec{\nabla} f)$$

So $$\vec{\nabla} \times \left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) = -\vec{J}(\vec{r}') \times \vec{\nabla} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$$

$$-\vec{\nabla} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{where } \vec{\mathscr{r}} = \vec{r}-\vec{r}'$$

( ) is potential of charge with $q=1$ — electric field of charge with $q=1$ — $-\vec{\nabla} V = \vec{E}$

So $$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}$$

This is just the Biot - Savart Law!

Now apply to our problem of finding $\vec{E}$ when $\rho = 0$ but $\frac{\partial \vec{B}}{\partial t} \neq 0$

Gauss $\vec{\nabla} \cdot \vec{E} = 0$ compare to $\vec{\nabla} \cdot \vec{B} = 0$

Faraday $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\vec{\nabla} \cdot \vec{B} = \mu_0 \vec{J}$

solution for $\vec{E}$ obtained from our magnetostatic solution by making the substitutions

$$\vec{B} \to \vec{E}, \quad \mu_0 \vec{J} \to -\frac{\partial \vec{B}}{\partial t}$$

So

$$\boxed{\vec{E}(\vec{r},t) = \frac{-1}{4\pi} \int d^3r' \left(\frac{\partial \vec{B}(\vec{r}',t)}{\partial t}\right) \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}$$

$$= -\frac{\partial}{\partial t} \left[\frac{1}{4\pi} \int d^3r' \, \vec{B}(\vec{r}',t) \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \right]$$

So if we know $\vec{B}(\vec{r},t)$ we can find the induced $\vec{E}(\vec{r},t)$.

You may have seen the Biot-Savart Law just for the case of a current carrying wire, where $\vec{J} \neq 0$ only along the path of a one dimensional curve.

In that case $d^3r' \, \vec{J}(\vec{r}') = d\vec{l}' \, I$

↑ differential tangent to curve

and so

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int d\vec{l}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$