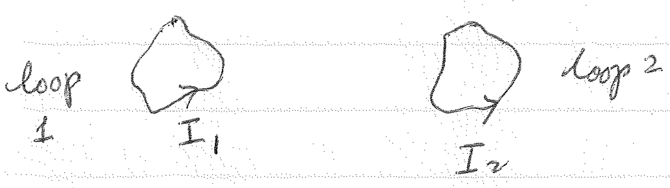
energy stored up in configuration of current carrying loops



Method 1

What is the work done to create configuration of the two loops above?

1) move loops into position, with initially $I_1 = I_2 = 0$.
   when $I_1 = I_2 = 0$, the loops exert no force on each other, so this step takes no work.

2) Now turn up current in loop 1 to I_1
   " " " " " 2 to I_2

This step costs work as follows: As I_1 and I_2 change in time, they give rise to changing magnetic flux through the loops, which gives rise to emf's around the loops, which tends to oppose the ~~change in~~ currents I_1 and I_2. To keep the currents I_1 and I_2 flowing, we have to do work against this induced emf.

emf in loop 1: $\mathcal{E}_1 = \underbrace{-L_1 \frac{dI_1}{dt}}_{\text{self inductance}} \underbrace{- M \frac{dI_2}{dt}}_{\text{mutual inductance}}$

emf in loop 2: $\mathcal{E}_2 = \overbrace{-L_2 \frac{dI_2}{dt}} \overbrace{- M \frac{dI_1}{dt}}$

~~The emf is the work done per charge, moving charge around loop.~~

~~$\Delta Q \, \mathcal{E}$ is power dissipated by the emf~~

~~Δcharge/time~~

The emf $\mathcal{E}_1$ and $\mathcal{E}_2$ <u>oppose</u> the change in I_1 and I_2. To keep I_1 and I_2 steady, we have to ~~we~~ do work to <u>counter</u> this emf
~~to keep the steady current I flowing, we have to pump back in this power lost. So the total work done per unit time is~~

$$\frac{dW}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2$$

$\mathcal{E}$: work per charge in one trip around loop

I: charge per time traveling down wire

$$\frac{dW}{dt} = L_1 I_1 \frac{dI_1}{dt} + M I_1 \frac{dI_2}{dt} + L_2 I_2 \frac{dI_2}{dt} + M I_2 \frac{dI_1}{dt}$$

$$= \frac{1}{2} L_1 \frac{d(I_1^2)}{dt} + \frac{1}{2} L_2 \frac{d(I_2^2)}{dt} + M \frac{d(I_1 I_2)}{dt}$$

$$W = \int_0^T \frac{dW}{dt}\, dt = \underbrace{\frac{1}{2} L_1 I_1^2}_{\text{self energy of loop 1}} + \underbrace{\frac{1}{2} L_2 I_2^2}_{\text{self energy of loop 2}} + \underbrace{M I_1 I_2}_{\text{interaction energy between loops 1 + 2.}}$$

substitute in expressions for L_1, L_2, M

$$W = \frac{1}{2}\frac{\mu_0}{4\pi} I_1^2 \oint_1 \oint_1 \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r} - \vec{r}'|} + \frac{1}{2}\frac{\mu_0}{4\pi} I_2^2 \oint_2 \oint_2 \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r} - \vec{r}'|}$$

$$+ \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r} - \vec{r}'|}$$

$$\cancel{= \frac{1}{2}\frac{\mu_0}{4\pi} \left\{ \oint_1 \oint_1 dl\, dl' \frac{\vec{I} \cdot \vec{I}}{|\vec{r} - \vec{r}'|} \right.}$$

$$= \frac{1}{2}\frac{\mu_0}{4\pi} \left\{ \oint_1 \oint_1 dl\, dl' \frac{\vec{I}(\vec{r}) \cdot \vec{I}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \oint_2 \oint_2 dl\, dl' \frac{\vec{I}(\vec{r}) \cdot \vec{I}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right.$$

$$\left. + 2 \oint_1 \oint_2 dl\, dl' \frac{\vec{I}(\vec{r}) \cdot \vec{I}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right\}$$

where $\vec{I}(\vec{r})$ = current at position $\vec{r}$ on either loop 1 or loop 2 (depending on where $\vec{r}$ is)

$$W = \frac{1}{2}\frac{\mu_0}{4\pi}\oint_{1+2}\oint_{1+2} dl\,dl' \frac{\vec{I}(r)\cdot\vec{I}(r')}{|\vec{r}-\vec{r}'|}$$

generalization to n loops: $W = \frac{1}{2}\frac{\mu_0}{4\pi}\oint_{1+2+\cdots n}\oint_{1+2+\cdots n} dl\,dl' \frac{\vec{I}(\vec{r})\cdot\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

generalization to continuous current distribution:

$$W = \frac{1}{2}\frac{\mu_0}{4\pi}\iint d^3r\, d^3r' \frac{\vec{J}(\vec{r})\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ (compare to electrostatic energy $W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\iint d^3r\,d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$)

use $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$ in Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$

$$W = \frac{1}{2}\int d^3r\, \vec{A}(\vec{r})\cdot\vec{J}(\vec{r})$$ (compare to electrostatic energy $W = \frac{1}{2}\int d^3r\, V(\vec{r})\rho(\vec{r})$)

For a steady current, ie magnetostatic situation,

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\Rightarrow W = \frac{1}{2\mu_0}\int d^3r\, \vec{A}(\vec{r})\cdot\left[\vec{\nabla}\times\vec{B}(\vec{r})\right]$$ use $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

$$= \frac{1}{2\mu_0}\int d^3r\left[\vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B})\right]$$

$$= \frac{1}{2\mu_0}\int_{vol} d^3r\, B^2 - \frac{1}{2\mu_0}\oint_{surface}(\vec{A}\times\vec{B})\cdot d\vec{a}$$

as let volume ~~include~~ all of space, surface $\to\infty$, then for localized current source $\vec{J}$, the $\vec{B}$ and $\vec{A}$ will $\to 0$ as $\vec{r}\to\infty$ sufficiently fast that the surface integral will vanish.

$\Rightarrow W = \frac{1}{2\mu_0}\int d^3r\, B^2$ (compare to electostatic $W = \frac{1}{2}\epsilon_0\int d^3r\, E^2$)

Summary

magnetostatic energy

$$W = \frac{1}{2}\frac{\mu_0}{4\pi}\int d^3r \int d^3r' \, \frac{\vec{J}(\vec{r})\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$W = \frac{1}{2}\int d^3r \, \vec{A}(\vec{r})\cdot\vec{J}(\vec{r})$$

$$W = \frac{1}{2\mu_0}\int d^3r \, |\vec{B}(\vec{r})|^2$$

electrostatic energy

$$W = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\int d^3r \int d^3r' \, \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$W = \frac{1}{2}\int d^3r \, V(\vec{r})\rho(\vec{r})$$

$$W = \frac{1}{2}\varepsilon_0\int d^3r \, |\vec{E}(\vec{r})|^2$$

Method ②

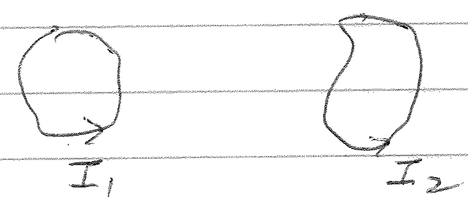
Alternative way to construct the configuration of current carrying loops

i) Start with loops infinitely far apart with zero current

ii) turn up currents to I_1 and I_2. This costs work $\frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2$ by previous arguments

iii) move loops in from infinity to final positions keeping I_1 and I_2 held constant.

What is the work done in step (iii)?



Let us move loop 1 into its final position, keeping loop 2 at infinity. This causes no work as the loops stay infinitely apart. Now move loop 2 into its final position. As move loop 2, there is a magnetostatic force acting on it from the field $\vec{B}_1$ produced by the current in loop 1.

Force on moving charge is $q\vec{v} \times \vec{B}$

Force on element of current density is $d^3r\, \vec{J}(\vec{r}) \times \vec{B}(\vec{r})$

Force on element of current in a wire is $I\, d\vec{\ell} \times \vec{B}$

Force on loop 2 due to magnetic field $\vec{B}_1$ of loop 1 is

$$\vec{F}_2 = I_2 \oint_2 d\vec{l}_2 \times \vec{B}_1(\vec{r}_2)$$

use Biot-Savart Law $\vec{B}_1(\vec{r}_2) = \frac{\mu_0 I_1}{4\pi} \oint_1 d\vec{l}_1 \times \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$

So

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l}_2 \times (d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1))}{|\vec{r}_2 - \vec{r}_1|^3}$$

Now use triple product rule $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$d\vec{l}_2 \times (d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1)) = d\vec{l}_1 \, (d\vec{l}_2 \cdot (\vec{r}_2 - \vec{r}_1)) - (\vec{r}_2 - \vec{r}_1)(d\vec{l}_1 \cdot d\vec{l}_2)$$

$$\vec{F}_2 = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \, d\vec{l}_2 \cdot \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

$$- \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \cdot d\vec{l}_2 \, \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

first term can be written as

$$- \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\vec{l}_1 \, d\vec{l}_2 \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right)$$

Now $\oint_2 d\vec{l}_2 \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right) = 0$ as line integral of any gradient around a closed loop vanishes

So $$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2) \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

Now let $\vec{r}_2 = \vec{R} + \delta\vec{r}_2$ where $\vec{R}$ is the center of loop 2. As loop 2 moves, $\vec{R}$ goes from infinity to its final position $\vec{R}_0$

$$\vec{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2) \frac{(\vec{R} + \delta\vec{r}_2 - \vec{r}_1)}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|^3}$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2)\, \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

derivative with respect to $\vec{R}$,

$\vec{F}_2$ is force acting on loop 2 from loop 1. To move loop 2 into position, the person moving loop 2 must exert a force equal and opposite to $\vec{F}_2$. The work done moving the loop is therefore

$$\tilde{W} = -\int_{\infty}^{\vec{R}_0} \vec{F}_2 \cdot d\vec{R} \qquad \text{(mover applies force } -\vec{F}_2\text{)}$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 (d\vec{l}_1 \cdot d\vec{l}_2) \underbrace{\int_{\infty}^{\vec{R}_0} d\vec{R} \cdot \vec{\nabla} \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)}$$

can easily do the line integral of a gradient

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\vec{l}_1 \cdot d\vec{l}_2)}{|\vec{R}_0 + \delta\vec{r}_2 - \vec{r}_1|}$$

$$\tilde{W} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|} \qquad \vec{r}_2 = \vec{R}_0 + \delta\vec{r}_2$$

compare with our derivation of the mutual inductance M and we see

$$\tilde{W} = -MI_1I_2$$

Thus this method appears to give as the work done to create the current carrying loops in their final position

method ② $$W = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 - MI_1I_2 \quad !!$$

Compare to our first method which gave

method ① $$W = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2$$

In method ①, the interaction energy of the loops was MI_1I_2

In method ②, it appears that the interaction energy of the two loops is $-MI_1I_2$

How do we reconcile these two answers?

The energy stored in the configuration of current carrying loops should <u>not</u> depend on the process used to construct the configuration if energy is conserved!