$\Rightarrow$ Maxwell's Equ only look simple when expressed in terms of <u>Fourier Transforms</u>.

For pure sinusoidal solutions: $\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{B}(\vec{r},t) = \vec{B}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{H}(\vec{r},t) = \vec{H}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\vec{D}(\vec{r},t) = \vec{D}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

for EM waves in dielectric, assume $\rho_f = \vec{J}_f = 0$

Maxwell's Equs: $\vec{\nabla}\cdot\vec{D}=0$, $\vec{\nabla}\cdot\vec{B}=0$, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$

assume $\mu=\mu_0 \Rightarrow \vec{H}_\omega = \frac{1}{\mu_0}\vec{B}_\omega$

dielectric response given by $\varepsilon(\omega) \Rightarrow \vec{D}_\omega = \varepsilon(\omega)\vec{E}_\omega$

<u>For $\rho_f=\vec{J}_f=0$</u>, Maxwell's Equs in terms of the Fourier amplitudes are then

$$\left.\begin{array}{llll}
1) & i\vec{k}\cdot\vec{D}_\omega = i\varepsilon(\omega)\,\vec{k}\cdot\vec{E}_\omega = 0 & \rightarrow \vec{k}\cdot\vec{E}_\omega = 0 & \vec{k}\perp\vec{E}_\omega \\
2) & i\vec{k}\cdot\vec{B}_\omega = 0 & \rightarrow \vec{k}\cdot\vec{B}_\omega = 0 & \vec{k}\perp\vec{B}_\omega
\end{array}\right\}\text{transverse}$$

3) Faraday $\quad i\vec{k}\times\vec{E}_\omega = i\omega\vec{B}_\omega$

4) Ampere $\quad i\vec{k}\times\vec{H}_\omega = -i\omega\vec{D}_\omega \Rightarrow \frac{i\vec{k}}{\mu_0}\times\vec{B}_\omega = -i\omega\varepsilon(\omega)\vec{E}_\omega$

$$\vec{k}\times(\text{Faraday}) = i\vec{k}\times(\vec{k}\times\vec{E}_\omega) = i\omega(\vec{k}\times\vec{B}_\omega)$$ substitute in from Ampere

$$= -i\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\vec{k}\times(\vec{k}\times\vec{E}_\omega) = \vec{k}\,\underbrace{(\vec{k}\cdot\vec{E}_\omega)}_{=0\text{ by (1)}} - \vec{E}_\omega(\vec{k}\cdot\vec{k}) = -\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Rightarrow k^2\vec{E}_\omega = \omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Rightarrow \boxed{\begin{aligned} k^2 &= \omega^2 \varepsilon(\omega)\mu_0 \\ k^2 &= \frac{\omega^2}{c^2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right)\end{aligned}}$$

use $\frac{1}{c^2} = \mu_0 \varepsilon_0$

"dispersion" relation for waves in dielectric

dispersion relation determines wave vector k, for a given frequency ω.

Note $\frac{\omega^2}{k^2} \neq$ constant $\Rightarrow \vec{E}$ is <u>not</u> solution of a wave equation $\Box^2 \vec{E} = 0$.

different frequencies travel with <u>different</u> speeds.

Since $\varepsilon(\omega)$ is complex $\quad \varepsilon(\omega) = \underbrace{\varepsilon_1(\omega)}_{Re[\varepsilon]} + i \underbrace{\varepsilon_2(\omega)}_{Im[\varepsilon]}$

then in general the wavevector is also complex

$$k = k_1 + i k_2 = \pm \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{\varepsilon_0} + i\frac{\varepsilon_2}{\varepsilon_0}}$$

For a wave traveling in $\hat{z}$ direction, $\vec{k} = k\hat{z}$, we have

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_\omega e^{i((k_1 + i k_2) z - \omega t)}$$

$$= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

If choose $+\sqrt{\ }$ solution for k_1, so that wave propagates in $+\hat{z}$ direction, then should take $+\sqrt{\ }$ solution for k_2, so that wave decays as it propagates into material decay length $= 1/k_2$ $\quad$ k_2 is called the attenuation

Since intensity is $\sim E^2$ decays as $e^{-2k_2 z}$, $2k_2$ is called the absorption coefficient

physical origin of decay: EM wave excits atom to oscillate. Oscillations pump energy into other degrees of freedom, due to damping γ. $\Rightarrow$ EM wave is pumping energy into material $\Rightarrow$ Energy contained in EM wave should decrease as it propagates into material $\Rightarrow$ amplitude decays.

phase velocity of wave $\quad v_p \equiv \dfrac{\omega}{k_1}$ $\quad$ depends on frequency

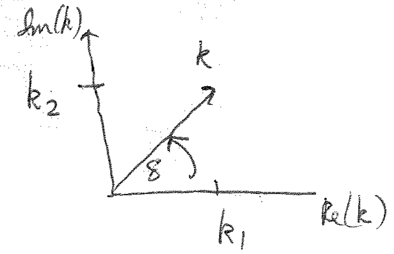
index of refraction $\quad n \equiv \dfrac{c}{v_p} = \dfrac{ck_1}{\omega}$ $\quad$ depends on freq

Lets look now at magnetic field. From Faraday

$$\vec{B}_\omega = \frac{\vec{k}}{\omega} \times \vec{E}_\omega = \frac{(k_1 + ik_2)}{\omega}\, \hat{z} \times \vec{E}_\omega$$

write $k_1 + ik_2 = \sqrt{k_1^2 + k_2^2}\, e^{i\delta}$ $\quad$ where $\delta = \arctan\left(\dfrac{k_2}{k_1}\right)$

$\qquad = |k| e^{i\delta}$ $\quad$ is phase of k



$$\vec{B}_\omega = \frac{|k|}{\omega}\, \hat{z} \times \vec{E}_\omega\, e^{i\delta}$$

$$\vec{B}(\vec{r},t) = \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta)}$$

$$= \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z-\omega t+\delta)}$$

<u>Physical fields</u>:

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2 z}\cos(k_1 z-\omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\,\frac{|k|}{\omega}\, e^{-k_2 z}\cos(k_1 z-\omega t+\delta)$$

→ (1) $\vec{E}$ and $\vec{B}$ are transverse to $\vec{k}$, and $\vec{E}\perp\vec{B}$

(2) ratio of amplitudes $\frac{|\vec{B}|}{|\vec{E}|} = \frac{|k|}{\omega} = \frac{\sqrt{k_1^2+k_2^2}}{\omega} = \sqrt{\left|\frac{\varepsilon(\omega)}{\varepsilon_0}\right|}\,\frac{1}{c}$

(3) $\vec{B}$ wave is shifted with respect to $\vec{E}$ wave by phase shift $\delta = \arctan(k_2/k_1)$ (see Fig 8.21 in text)

<u>Summary</u>

Main consequences of <u>complex</u> $\varepsilon(\omega)$

1) Waves decay as they propagate $\sim e^{-k_2 z}$

2) $\vec{E}$ and $\vec{B}$ waves shifted in phase by $\delta = \arctan(k_2/k_1)$

If $\varepsilon_2 = \mathrm{Im}[\varepsilon(\omega)] = 0$, then $\underbrace{\varepsilon \text{ real}, \Rightarrow k \text{ real}}_{\text{and if } \varepsilon_1>0}$, $k_2 = 0$

⇒ no decay and no phase shift.

Main consequences of <u>freq dependent</u> $\varepsilon(\omega)$

(1) $\vec{E}(t)$ and $\vec{D}(t)$ non-locally related in time

(2) waves of different ω travel with different velocities $v_p = \frac{\omega}{k_1}$

(3) <u>dispersion</u> – wave pulses do not travel with v_p, and do not keep their shape as they propagate

Phase velocity and group velocity and dispersion

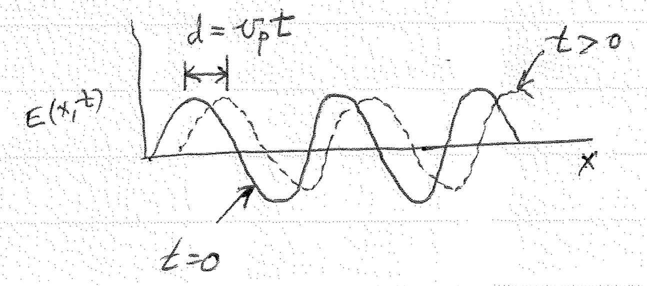
$k^2 = \frac{\omega^2}{c^2}\frac{\varepsilon(\omega)}{\varepsilon_0}$ For simplicity, assume $\varepsilon(\omega)$ is real and positive

$k = \frac{\omega}{c}\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$ $v_p = \frac{\omega}{k} = c\sqrt{\frac{\varepsilon_0}{\varepsilon(\omega)}} = \frac{c}{n}$

index of refraction $n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} = \sqrt{K(\omega)}$ dielectric function

sinusoidal waves $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ propagate with different phase speeds $v_p(\omega)$ for different ω.

v_p is speed with which peaks in oscillation move to right



for $\vec{E} = E e^{i(kx-\omega t)}$
with $\omega = v_p(\omega)\, k$

If take linear superposition of many sinusoidal waves, then each different freq ω, moves with different speed $v_p(\omega)$. So the shape of the wave is <u>not</u> preserved in time.

[This is another way to see that waves in a dielectric do not solve the wave equation - for the wave equation, all freq move with same speed v indep of ω, and the shape of the wave is always preserved in time, ie solutions are always of form $g(\vec{k}\cdot\vec{r}-\omega t)$]

Consider a superposition

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)} \qquad k(\omega) = \frac{\omega}{c}\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

At $\vec{r}=0$, $\vec{E}(0,t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega t}$ so $\vec{E}_\omega$ is F.T. of $\vec{E}(0,t)$

At some position $\vec{r} \neq 0$

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)}$$

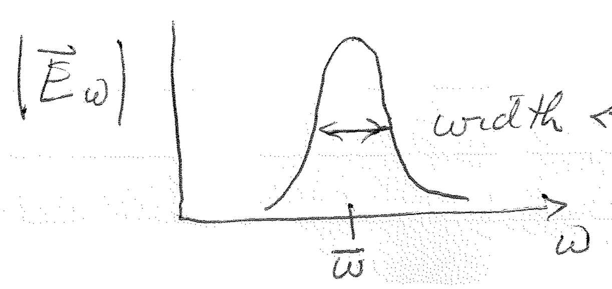
if <u>no</u> dispersion, ie $k = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}} = \frac{\omega}{v_p}$ with v_p indep of ω

Then $\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - z/v_p)}$

$= \vec{E}(0, t - z/v_p)$ $\leftarrow$ form of solution to wave equation

field at z at time t, is same field as was at $z=0$ at the earlier time $t - z/v_p$ $\Rightarrow$ wave moved distance z in time z/v_p $\Rightarrow$ speed of wave is v_p

Suppose now that $\varepsilon(\omega)$ <u>does</u> depend on ω, so there is dispersion. Suppose $\vec{E}_\omega$ is strongly peaked about some average $\bar{\omega}$



then $k(\omega) \cong k(\bar{\omega}) + \left.\frac{dk}{d\omega}\right|_{\bar{\omega}} (\omega - \bar{\omega}) + \cdots$

$$\vec{E}(\vec{r},t) = \int d\omega\, \vec{E}_\omega\, e^{i(k(\bar{\omega})z + \frac{dk}{d\omega}\omega z - \frac{dk}{d\omega}\bar{\omega} z - \omega t)}$$

$$= e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z} \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - \frac{dk}{d\omega} z)}$$

$$= \underbrace{e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z}}_{\text{phase factor}} \; \underbrace{\vec{E}(0, t - \frac{dk}{d\omega} z)}_{\text{envelope - determines shape of pulse}}$$

intensity of wave $\sim |\vec{E}|^2$

$$|\vec{E}|^2(\vec{r},t) = |\vec{E}|^2(0, t - \frac{dk}{d\omega} z)$$

intensity travels with velocity $v_g = 1 \Big/ \left(\frac{dk}{d\omega}\right)_{\bar{\omega}} = \frac{d\omega}{dk} \equiv$ group velocity

not with average phase velocity $\bar{v}_p = \frac{\bar{\omega}}{k(\bar{\omega})}$

only when $\varepsilon(\omega)$ is indep of ω will $v_p = v_g$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c} n(\omega)\right] = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\,\omega\,\frac{dn}{d\omega}} \Rightarrow \text{when } \frac{dn}{d\omega} > 0,\ v_g < v_p \quad (1)$$

$$\text{when } \frac{dn}{d\omega} < 0,\ v_g > v_p \quad (2)$$

case (1) is called "normal" dispersion

case (2) is called "anomalous" dispersion