So far, all our results come from the requirement that the phases of the incident, reflected, and transmitted waves all match at the interface. This is enough to determine the directions, wavelengths, attenuation, and frequencies of the waves. These results hold for <u>any</u> type of wave, not just electromagnetic waves.

Now want to solve for amplitudes of transmitted and reflected waves.

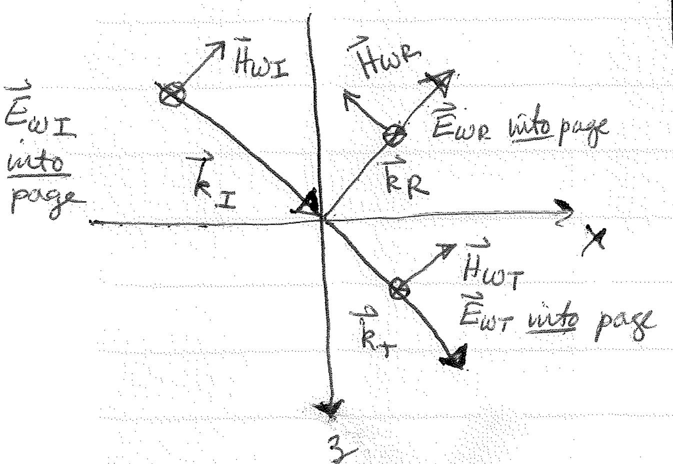
<u>two cases:</u> "plane of incidence" = plane spanned by the wavevectors $\vec{k}_I$, ~~[illegible]~~ and the normal to the interface — in our case, the xz plane

① $\vec{E}_\omega$ is $\perp$ to the plane of incidence

② $\vec{E}_\omega$ is $\parallel$ to the plane of incidence

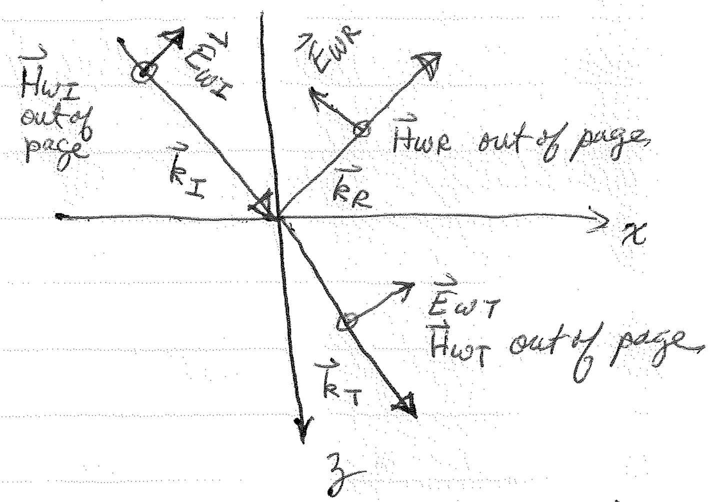
the most general case is a linear superposition of these two, so treating these two cases separately also gives the general solution.

$E_0 \perp$ plane of incidence



($H_{\omega I}$ in plane of incidence)

$E_0 \parallel$ plane of incidence



($H_{\omega I} \perp$ to plane of incidence)

continuity of $\hat{y}$ components of $\vec{E}$ / of $\vec{H}$

all the $\vec{E}$'s are along $\hat{y}$

(1) $E_I + E_R = E_T$

where $\vec{E}_{\omega I} = E_I \hat{y}$ etc.

all the $\vec{H}$'s are along $\hat{y}$

(1) $H_I + H_R = H_T$

where $\vec{H}_{\omega I} = H_I \hat{y}$ etc.

continuity of $\hat{x}$ components of $\vec{H}$ / of $\vec{E}$

$H_{Ix} + H_{Rx} = H_{Tx}$

Faraday $H_x = \dfrac{k_z}{\omega\mu} E_y$
$i\mu\omega\vec{H} = i\vec{k}\times\vec{E}$

plug in above and use $k_{Iz} = -k_{Rz}$

$\Rightarrow$

(2) $\dfrac{k_{Iz}}{\mu_a}(E_I - E_R) = \dfrac{k_{Tz}}{\mu_b} E_T$

solve equation (1) and (2) for E_R and E_T in terms of E_I

$E_{Ix} + E_{Rx} = E_{Tx}$

Ampere $E_x = \dfrac{-k_z}{\omega\varepsilon} H_y$
$-i\omega\varepsilon\vec{E} = i\vec{k}\times\vec{H}$

plug in above and use $k_{Iz} = -k_{Rz}$

$\Rightarrow$

(2) $\dfrac{k_{Iz}}{\varepsilon_a}(H_I - H_R) = \dfrac{k_{Tz}}{\varepsilon_b} H_T$

solve equations (1) and (2) for H_R and H_T in terms of H_I

$$E_R = \frac{\mu_b k_{Iz} - \mu_a k_{Tz}}{\mu_b k_{Iz} + \mu_a k_{Tz}} E_I$$

$$E_T = \frac{2\mu_b k_{Iz}}{\mu_a k_{Tz} + \mu_b k_{Iz}} E_I$$

$$H_R = \frac{\varepsilon_b k_{Iz} - \varepsilon_a k_{Tz}}{\varepsilon_b k_{Iz} + \varepsilon_a k_{Tz}} H_I$$

$$H_T = \frac{2\varepsilon_b k_{Iz}}{\varepsilon_a k_{Tz} + \varepsilon_b k_{Iz}} H_I$$

We can now define the reflection and transmission coefficients.
These are defined in terms of the transported energy.
Since the energy flux $|\vec{S}|$ is $\sim |\vec{E}|^2 \sim |\vec{H}|^2$, we have

<u>Reflection coefficient</u>

① $E_0 \perp$ to plane of incidence

$$R_\perp = \frac{|E_R|^2}{|E_I|^2} = \left| \frac{\mu_b k_{Iz} - \mu_a k_{Tz}}{\mu_b k_{Iz} + \mu_a k_{Tz}} \right|^2$$

② $E_0 \parallel$ to plane of incidence

$$R_\parallel = \frac{|H_R|^2}{|H_I|^2} = \left| \frac{\varepsilon_b k_{Iz} - \varepsilon_a k_{Tz}}{\varepsilon_b k_{Iz} + \varepsilon_a k_{Tz}} \right|^2$$

For region of "total reflection" in material b, $\operatorname{Im} \varepsilon_b \simeq 0$, $\operatorname{Re} \varepsilon_b < 0$

$\Rightarrow \vec{k}_T \equiv i \vec{\kappa}_T$ where $\vec{\kappa}_T$ is real ($\vec{k}_T$ is pure imaginary)

$$\Rightarrow R_\perp = \left| \frac{\mu_b k_{Iz} - i\mu_a \kappa_{Tz}}{\mu_b k_{Iz} + i\mu_a \kappa_{Tz}} \right|^2$$

$$R_\parallel = \left| \frac{\varepsilon_b k_{Iz} - i\varepsilon_a \kappa_{Tz}}{\varepsilon_b k_{Iz} + i\varepsilon_a \kappa_{Tz}} \right|^2$$

both are of the form $\left| \frac{a - ib}{a + ib} \right|^2 = 1$ when a, b both real

$\Rightarrow R_\perp = R_\parallel = 1$ ~~when material~~ b

this confirms that material b is totally reflecting in this region of frequency

When medium b is transparent, ie ε_b is real and $\varepsilon_b > 0$ we have

$$k_{Iz} = \omega\sqrt{\mu_a\varepsilon_a}\cos\theta_I = \frac{\omega}{c} n_a \cos\theta_I$$
$$k_{Tz} = \omega\sqrt{\mu_b\varepsilon_b}\cos\theta_T = \frac{\omega}{c} n_b \cos\theta_T$$

and Snell's Law applies, so $n_a \sin\theta_I = n_b \sin\theta_T \Rightarrow \frac{n_b}{n_a} = \frac{\sin\theta_I}{\sin\theta_T}$

we can now write $R_\perp$ and $R_\parallel$ as functions of θ_I

For simplicity take $\mu_a = \mu_b = \mu_0$

1. $$R_\perp = \left(\frac{n_a\cos\theta_I - n_b\cos\theta_T}{n_a\cos\theta_I + n_b\cos\theta_T}\right)^2 = \left(\frac{\cos\theta_I - \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T}\right)^2$$

$$= \left(\frac{\sin\theta_T\cos\theta_I - \sin\theta_I\cos\theta_T}{\sin\theta_T\cos\theta_I + \sin\theta_I\cos\theta_T}\right)^2 = \left(\frac{\sin(\theta_I - \theta_T)}{\sin(\theta_I + \theta_T)}\right)^2$$

~~[illegible]~~

for $\theta_I = 0$, ie normal incidence, $\theta_I = \theta_T = 0$

$$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$$ if $n_a = n_b$, no reflection!

(2) $$R_{\parallel} = \left(\frac{\varepsilon_b n_a \cos\theta_I - \varepsilon_a n_b \cos\theta_T}{\varepsilon_b n_a \cos\theta_I + \varepsilon_a n_b \cos\theta_T}\right)^2$$

use $\sqrt{\varepsilon_b \mu_0} = \frac{n_b}{c}$

$\Rightarrow \varepsilon_b = \frac{n_b^2}{c^2\mu_0} = n_b^2 \varepsilon_0$

$\varepsilon_a = n_a^2 \varepsilon_0$

$$= \left(\frac{n_b \cos\theta_I - n_a \cos\theta_T}{n_b \cos\theta_I + n_a \cos\theta_T}\right)^2$$

$$= \left(\frac{\cos\theta_I - \left(\frac{\sin\theta_T}{\sin\theta_I}\right)\cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_T}{\sin\theta_I}\right)\cos\theta_T}\right)^2 = \left(\frac{\sin\theta_I \cos\theta_I - \sin\theta_T \cos\theta_T}{\sin\theta_I \cos\theta_I + \sin\theta_T \cos\theta_T}\right)^2$$

$$R_{\parallel} = \left(\frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)}\right)^2$$ $\Leftarrow$ after some algebra

for $\underline{\underline{\theta_I = 0}}$ $\Rightarrow$ $\theta_T = 0$

$$R_{\parallel} = \left(\frac{\varepsilon_b n_a - \varepsilon_a n_b}{\varepsilon_b n_a + \varepsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2$$

same as for $R_\perp$! But when $\theta_I = 0$ there is no distinction between the two case $\perp$ and $\parallel$, so this is to be expected.

When $\theta_I + \theta_T = \pi/2$, then $\tan(\theta_I + \theta_T) \to \infty$ and $R_{\parallel} = 0$.

This occurs at an angle of incidence $\theta_I \equiv \theta_B$ "Brewster's angle"

θ_B determined by $\underset{\theta_I}{\underset{\parallel}{n_a \sin\theta_B}} = n_b \sin\underbrace{\left(\frac{\pi}{2} - \theta_B\right)}_{=\theta_T} = n_b \cos\theta_B$

$$\Rightarrow \boxed{\tan\theta_B = \frac{n_b}{n_a}}$$

For a wave incident at θ_B, the reflected wave will always have $\vec{E}_R \perp$ plane of incidence, no matter what orientation of incoming $\vec{E}_I$, since $R_{\parallel} = 0$. ~~That is only $R_{\perp} \neq 0$, so reflected wave can only have $\vec{E} \perp$ plane of incidence~~.
If incoming wave has component of $\vec{E}_I \parallel$ to plane of incidence, this component gets purely transmitted since $R_{\parallel} = 0$.
Only the component of $\vec{E}_I \perp$ to plane of incidence can get reflected, since $R_{\perp} \neq 0$. $\Rightarrow$ reflected wave is polarized with $\vec{E}_R \perp$ to plane of incidence.

Generally, for all θ_I close to θ_B, $R_{\parallel} \ll R_{\perp}$, and the reflected wave is strongly polarized with $\vec{E}_R$ mostly $\perp$ to plane of incidence.

This is therefore one method to create a polarized light wave.

Additional notes on Reflection & Transmission Coefficients

For a transparent medium, the energy current can be written as (see text 9.3.1)

$$\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) = \vec{E}\times\vec{H} \qquad (\text{in a vacuum } \mu=\mu_0)$$

For a plane wave $\vec{E}(\vec{r},t) = \vec{E}_\omega \cos(\vec{k}\cdot\vec{r}-\omega t) \qquad \vec{E}_\omega \perp \vec{k}$

From lecture 13 we have

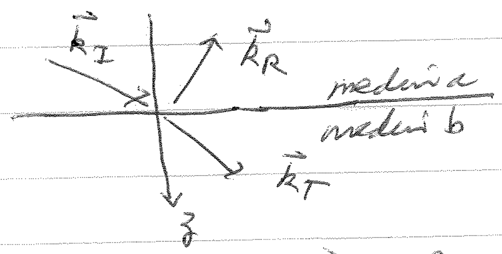
$$\vec{H}(\vec{r},t) = \frac{\vec{B}(\vec{r},t)}{\mu} = (\hat{k}\times\vec{E}_\omega)\frac{|k|}{\omega\mu}\cos(\vec{k}\cdot\vec{r}-\omega t)$$

(since the medium is transparent, k is real and $k_2=0$, $\delta = \arctan\frac{k_2}{k_1} = 0$)

$$\Rightarrow \quad \vec{S} = \vec{E}\times\vec{H} = \frac{|k|}{\omega\mu}\underbrace{\vec{E}_\omega\times(\hat{k}\times\vec{E}_\omega)}_{=|\vec{E}_\omega|^2\hat{k} \text{ since } \hat{k}\perp\vec{E}_\omega}\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

So

$$\langle\vec{S}\rangle = \frac{|k|}{2\omega\mu}|\vec{E}_\omega|^2\hat{k} \qquad \text{as } \langle\cos^2(\vec{k}\cdot\vec{r}-\omega t)\rangle = 1/2$$



The energy flux of the incident wave going into medium b is

$$\langle\vec{S}_I\rangle\cdot\hat{z} = \frac{|k_I|}{2\omega\mu_a}|\vec{E}_{I\omega}|^2(\hat{k}_I\cdot\hat{z}) = \frac{|k_I|}{2\omega\mu_a}|\vec{E}_{I\omega}|^2\cos\theta_I$$

The energy flux of the reflected wave is

use $\theta_I = \theta_R$

$$\langle\vec{S}_R\rangle\cdot\hat{z} = \frac{|k_R|}{2\omega\mu_a}|\vec{E}_{R\omega}|^2(\hat{k}_R\cdot\hat{z}) = \frac{|k_R|}{2\omega\mu_a}|\vec{E}_{R\omega}|^2(-\cos\theta_I)$$

(−) because reflected back

The energy flux of the transmitted wave is

$$\langle\vec{S}_T\rangle\cdot\hat{z} = \frac{|k_T|}{2\omega\mu_b}|\vec{E}_{T\omega}|^2(\hat{k}_T\cdot\hat{z}) = \frac{|k_T|}{2\omega\mu_b}|\vec{E}_{T\omega}|^2\cos\theta_T$$

Energy in = Energy out

$$\Rightarrow \langle \vec{S}_I \rangle \cdot \hat{z} + \langle \vec{S}_R \rangle \cdot \hat{z} = \langle \vec{S}_T \rangle \cdot \hat{z}$$

or $\langle \vec{S}_I \rangle \cdot \hat{z} = \langle \vec{S}_T \rangle \cdot \hat{z} - \langle \vec{S}_R \rangle \cdot \hat{z}$

this term is > 0 this term is < 0

$$|\langle \vec{S}_I \rangle \cdot \hat{z}| = |\langle \vec{S}_T \rangle \cdot \hat{z}| + |\langle \vec{S}_R \rangle \cdot \hat{z}|$$

If we define $R = \dfrac{|\langle \vec{S}_R \rangle \cdot \hat{z}|}{|\langle \vec{S}_I \rangle \cdot \hat{z}|}$, $T = \dfrac{|\langle \vec{S}_T \rangle \cdot \hat{z}|}{|\langle \vec{S}_I \rangle \cdot \hat{z}|}$

Then we get

$$1 = T + R$$

Also

$$R = \frac{\frac{|k_R|}{2\omega\mu_a} |\vec{E}_{R\omega}|^2 \cos\theta_I}{\frac{|k_I|}{2\omega\mu_a} |\vec{E}_{I\omega}|^2 \cos\theta_I} = \frac{|\vec{E}_{R\omega}|^2}{|\vec{E}_{I\omega}|^2}$$ since $|k_I| = |k_R|$

But

$$T = \frac{\frac{|k_T|}{2\omega\mu_b} |\vec{E}_{T\omega}|^2 \cos\theta_T}{\frac{|k_I|}{2\omega\mu_a} |\vec{E}_{I\omega}|^2 \cos\theta_I} \neq \frac{|\vec{E}_{T\omega}|^2}{|\vec{E}_{I\omega}|^2}$$

Radiation from moving charges

In Lorentz gauge: $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$

potentials solve $\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \equiv \Box^2 V = -\rho/\varepsilon_0$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t} \equiv \Box^2\vec{A} = -\mu_0\vec{J}$$

if we know potentials, can get fields from

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad , \quad \vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

As in electro and magneto statics, it is easier to solve for V and $\vec{A}$ and then determine $\bar{E}$ and $\vec{B}$, rather than try to solve for $\vec{E}$ and $\vec{B}$ directly.

Recall solutions for statics: $\nabla^2 V = -\rho/\varepsilon_0$, $\nabla^2\vec{A} = -\mu_0\vec{J}$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(r')}{|r-r'|} \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')}{|r-r'|}\,d^3r'$$

Both solutions follow from the fact that

$$\nabla^2\left(\frac{1}{|r-r'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}\,')$$

↑ Dirac δ- function

$\frac{1}{|r-r'|}$ is called the "Green's function" for the operator ∇^2