Now evaluate $\vec{I}_1$ and $\vec{I}_2$

<u>i^{th} component of $\vec{I}_1$</u>

$I_{1i} = \int d^3r' \, j_i(\vec{r}',\omega) = -\int d^3r' \, r_i' \, \vec{\nabla}' \cdot \vec{j}$ via integration by parts
boundary term vanishes as $\vec{j} \to 0$ far from source

to see this use: trick ① $\vec{\nabla} \cdot (f\vec{g}) - \vec{g} \cdot \vec{\nabla} f = f \vec{\nabla} \cdot \vec{g}$ product rule
apply to right hand side with $f \equiv r_i'$, $\vec{g} \equiv \vec{j}$

$$\Rightarrow -\int d^3r' \, r_i' \, \vec{\nabla}' \cdot \vec{j} = -\int d^3r' \left\{ \vec{\nabla}' \cdot (r_i' \vec{j}) - \vec{j} \cdot \vec{\nabla}' r_i' \right\}$$

first term: trick ② $\int d^3r' \, \vec{\nabla}' \cdot (r_i' \vec{j}) = \underset{\text{surface} \to \infty}{\oint} d\vec{a} \cdot (r_i' \vec{j}) = 0$ as $\vec{j}(r' \to \infty) = 0$
as $\vec{j}$ is localized

second term: trick ③ $\vec{\nabla}' r_i' =$ unit vector in direction i
to see this, consider example: $\vec{\nabla} x = \frac{\partial x}{\partial x} \hat{x} + \frac{\partial x}{\partial y} \hat{y} + \frac{\partial x}{\partial z} \hat{z}$
$= \hat{x} + 0 + 0$

$$\Rightarrow \vec{j} \cdot \vec{\nabla}' r_i' = j_i$$

put together: $-\int d^3r' \, r_i' \, \vec{\nabla}' \cdot \vec{j} = 0 + \int d^3r' \, j_i = I_{1i}$ as desired

Now use trick ④ charge conservation $\Rightarrow \vec{\nabla}' \cdot \vec{j} = -\frac{\partial \rho}{\partial t} = i\omega \rho(\vec{r},\omega)$
since $\rho(\vec{r},t) = \rho(\vec{r},\omega) e^{-i\omega t}$

$$\vec{I}_1 = -\int d^3r' \, \vec{r}' \, \vec{\nabla}' \cdot \vec{j} = -i\omega \underbrace{\int d^3r' \, \vec{r}' \rho(\vec{r}',\omega)}_{\vec{p}(\omega) \text{ electric dipole moment}}$$

$$\boxed{\vec{I}_1 = -i\omega \vec{p}(\omega)}$$

i^{th} component of $\vec{I}_2$

$$I_{2i} = \int d^3r' \, (\hat{r}\cdot\vec{r}\,')\, j_i = \int d^3r' (\hat{r}\cdot\vec{r}\,')\,(\vec{j}\cdot\vec{\nabla}' r_i') \quad \text{by trick ⑤}$$

$$= \sum_{k=1}^{3} \hat{r}_k \int d^3r' \,(r_k' \,\vec{j})\cdot\vec{\nabla}' r_i' \qquad \text{writing out } \hat{r}\cdot\vec{r}\,' = \sum_{k=1}^{3} \hat{r}_k r_k' \text{ as sum over components}$$

$$= \sum_{k=1}^{3} \hat{r}_k \int d^3r' \left\{ \underbrace{\vec{\nabla}'\cdot(r_i' r_k' \vec{j})}_{=0 \text{ by trick ②}} - r_i' \,\vec{\nabla}'\cdot(r_k'\vec{j}) \right\} \qquad \text{by trick ① with } \begin{cases} f \equiv r_i' \\ \vec{g} \equiv r_k' \vec{j} \end{cases}$$

$$= -\sum_{k=1}^{3} \hat{r}_k \int d^3r' \left[\underbrace{r_i' r_k' \,\vec{\nabla}'\cdot\vec{j}}_{\substack{i\omega r_i' r_k' \rho \\ \text{by trick ④}}} + \underbrace{r_i' \,\vec{j}\cdot\vec{\nabla}' r_k'}_{\substack{r_i' j_k \\ \text{by trick ⑤}}} \right] \qquad \text{expanding } \vec{\nabla}\cdot(r_k'\vec{j}) \text{ as in trick ①}$$

$$= -\sum_{k=1}^{3} \hat{r}_k \int d^3r' \left[r_i' j_k + i\omega r_i' r_k' \rho \right]$$

<u>Last trick:</u> $\quad I_{2i} = \frac{1}{2} I_{2i} + \frac{1}{2} I_{2i}$

$$= \frac{1}{2}\sum_k \hat{r}_k \Big[\underbrace{\int d^3r' \, r_k' j_i}_{\text{from definition of } I_{2i}} - \underbrace{\int d^3r' \{ r_i' j_k + i\omega r_i' r_k' \rho \}}_{\text{from above manipulations}} \Big]$$

$$\vec{I}_2 = \frac{1}{2}\int d^3r' \Big[\underbrace{(\hat{r}\cdot\vec{r}\,')\vec{j} - (\hat{r}\cdot\vec{j})\vec{r}\,'}_{-\hat{r}\times(\vec{r}\,'\times\vec{j}) \text{ triple product rule}} \Big] - \frac{1}{2}\int d^3r' \, i\omega \, \hat{r}\cdot\vec{r}\,'\,\vec{r}\,'\rho$$

$$= -\frac{1}{2}\hat{r}\times\int d^3r' (\vec{r}\,'\times\vec{j}) - \frac{1}{2} i\omega\, \hat{r}\cdot\int d^3r' \,(\vec{r}\,'\vec{r}\,')\rho$$

$$= -\hat{r}\times\vec{m}(\omega) - \frac{1}{2}\frac{i\omega}{3}\,\hat{r}\cdot\overleftrightarrow{Q}'(\omega)$$

where $\vec{m}(\omega) \equiv \frac{1}{2}\int d^3r' \left(\vec{r}' \times \vec{J}(\vec{r}',\omega)\right)$ is magnetic dipole moment

$$\overleftrightarrow{Q}'_{ij}(\omega) = \int d^3r' \; 3\vec{r}_i{}'\vec{r}_j{}' \rho(\vec{r}';\omega)$$

looks very close to electric quadrapole tensor

$$\overleftrightarrow{Q}_{ij} = \int d^3r' \left(3\vec{r}_i{}'\vec{r}_j{}' - r'^2\delta_{ij}\right)\rho(\vec{r}',\omega)$$

$$\overleftrightarrow{Q}'_{ij} = \overleftrightarrow{Q}_{ij} + \delta_{ij}\int d^3r' \; r'^2 \rho(\vec{r}';\omega)$$

$$\vec{I}_2 = -\hat{r}\times\vec{m}(\omega) - \frac{i\omega}{6}\hat{r}\cdot\overleftrightarrow{Q}(\omega) - \frac{i\omega}{6}\hat{r}\underbrace{\int d^3r' \; r'^2\rho(\vec{r}';\omega)}_{\text{call this } C(\omega) \text{ a scalar}}$$

plug back into $\vec{A}(\vec{r},\omega)$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{-i\omega\vec{p} - \left(\frac{1}{r} - ik\right)\left(\hat{r}\times\vec{m} + \frac{i\omega}{6}\hat{r}\cdot\overleftrightarrow{Q} + \frac{i\omega}{6}\hat{r}\,C\right)\right\}$$

electric dipole contribution

magnetic dipole contribution

electric quadrapole contribution

The last piece which contributes to $\vec{A}$, ie $\frac{i\omega}{6} c\,\hat{r}\,\frac{e^{ikr}}{r}$ is unimportant – it does not effect the $\vec{E}$ or $\vec{B}$ fields since

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{\nabla}\times\left[f(r)\,\hat{r}\right] = 0$$

similarly, away from sources, where $\vec{J} = 0$, Ampere's law gives

$$\mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

$$-i\omega\mu_0\varepsilon_0\,\vec{E}(\vec{r},\omega) = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

since last term doesn't contribute to $\vec{B}$, it doesn't contribute to $\vec{E}$. Formally, we could remove it by making a gauge transformation. Less formally, we will just drop it!

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{-i\omega\vec{p} - \overbrace{\left(\frac{1}{r} - ik\right)}^{= -\left(1+\frac{i}{kr}\right)ik}\left(\hat{r}\times\vec{m} + \frac{i\omega}{6}\,\hat{r}\cdot\overleftrightarrow{Q}\right)\right\}$$

lets look at relative strengths of the different terms

far from sources, $\frac{1}{r}$ will be small compared to k.

radiation zone: just consider those terms in $\vec{A}$ that decrease as slowest powers of $\left(\frac{1}{r}\right)^n$. This will be the $\frac{1}{r}$ terms

Approx ① $d \ll r$

Approx ② $d \ll \lambda$

Radiation Zone $\lambda \ll r$ so $kr \gg 1$

Combine: $d \ll \lambda \ll r$ is RZ

lectric dipole term $\quad \vec{p} \simeq q\,d \qquad$ q is typical charge in source
d is size of source region

magnetic dipole term $\quad \vec{m} = \frac{1}{2}\int d^3r\ \vec{r}\times\vec{j} \qquad \vec{j} \sim qv$ where v is typical velocity

$$\simeq d\,j \simeq d\,v\,q \qquad v \sim \frac{d}{\tau} \sim d\omega \sim dck$$

$$\simeq q\,d^2\omega \sim q\,c\,d^2 k$$

electric quadrapole term $\quad \overleftrightarrow{Q} \sim \int d^3r\ \vec{r}\,\vec{r}\,\rho$

$$\sim q\,d^2$$

so electric dipole contrib to $\vec{A}$ goes as $\quad \omega\vec{p} \sim q\omega d = qc(kd)$
magnetic dipole contrib to $\vec{A}$ goes as $\quad k\vec{m} \sim q\omega k d^2 = qc(kd)^2$
electric quadrapole contrib to $\vec{A}$ goes as $\quad k\omega\overleftrightarrow{Q} \sim q\omega k d^2 = qc(kd)^2$

Since approx (2) assumed (kd) was small
(non relativistic approx : $kd \approx v/c$)
we have an expansion for $\vec{A}$ in powers of (kd)

leading term is the electric dipole term.
next order terms are $\left.\begin{cases}\text{magnetic dipole} \\ \text{electric quadrapole}\end{cases}\right\}$ ← these are comparable in strength.

If we kept higher order terms in our expansion, the next terms would be the magnetic quadrapole and electric octopole, both of order $qc(kd)^3$.

Consider now the leading term, the electric dipole term.

$$\vec{A}_{E1} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-i\omega) \vec{p}(\omega)$$

E1 ≡ electric dipole term

lets now compute the $\vec{E}$ and $\vec{B}$ fields in the electric dipole approx

<u>magnetic field</u>

$$\vec{B}_{E1}(\vec{r},\omega) = \vec{\nabla} \times \vec{A}_{E1}(\vec{r},\omega)$$

$$= -\frac{i\omega\mu_0}{4\pi} \vec{\nabla} \times \left(\frac{e^{ikr}}{r} \vec{p}(\omega) \right)$$

use $\vec{\nabla} \times (f\vec{g}) = f\,\vec{\nabla}\times\vec{g} + (\vec{\nabla} f) \times \vec{g}$

where $f = \frac{e^{ikr}}{r}$, $\vec{g} = \vec{p}$

$$= -\frac{i\omega\mu_0}{4\pi} \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) \times \vec{p}$$

use $\vec{\nabla} f(r) = \frac{\partial f}{\partial r} \hat{r}$

$$= -\frac{i\omega\mu_0}{4\pi} \left\{ \left(\frac{ik}{r} - \frac{1}{r^2} \right) e^{ikr}\, \hat{r} \times \vec{p} \right\}$$

use $\omega = ck$

$$= \frac{c\mu_0}{4\pi} k^2 \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr} \right) \hat{r} \times \vec{p}$$

$$\boxed{\vec{B}_{E1} = -\frac{c\mu_0}{4\pi} k^2 \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr} \right) \vec{p} \times \hat{r}}$$

ignore this as r gets large when $kr \gg 1$

ie in <u>radiation zone</u> limit:

$$\boxed{\vec{B}_{E1} = -\frac{c\mu_0}{4\pi} k^2 \frac{e^{ikr}}{r} \vec{p} \times \hat{r}}$$