Potentials from a moving point charge

Liénard - Wiechert potentials

In general

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \int dt' \, \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|} \, \delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \int dt' \, \frac{\vec{j}(\vec{r}',t')}{|\vec{r}-\vec{r}'|} \, \delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

For a pt charge moving on trajectory $\vec{r}_0(t)$ with velocity $\vec{v}(t) = \frac{d\vec{r}_0}{dt}$

$$\rho(\vec{r},t) = q\,\delta(\vec{r}-\vec{r}_0(t))$$
$$\vec{j}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t))$$

Substitute in

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \int dt' \, \frac{q\,\delta(\vec{r}'-\vec{r}_0(t'))}{|\vec{r}-\vec{r}'|} \, \delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

do $\int d^3r'$ using $\delta(\vec{r}'-\vec{r}_0(t'))$ term

$$= \frac{1}{4\pi\varepsilon_0}\int dt' \, \frac{q}{|\vec{r}-\vec{r}_0(t')|} \, \delta\left(t-t'-\frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)$$

now we want to do $\int dt'$ using the $\delta\left(t-t'-\frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)$

the δ-function is of the form $\delta(g(t'))$ where

$$g(t') = t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$

to evaluate, change variable of integration from t' to g

$$\int_{-\infty}^{\infty} dt' \, f(t') \, \delta(g(t')) = \int_{g(-\infty)}^{g(\infty)} f(t') \, \delta(g) \left(\frac{dt'}{dg}\right) dg$$

$$= f(t') \frac{dt'}{dg} = \frac{f(t')}{\left(\frac{dg}{dt'}\right)}$$ evaluated at t' such that $g(t') = 0$

Here: $f(t') = \dfrac{q}{|\vec{r} - \vec{r}_0(t')|}$

$$t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$ is the retarded time

$$\frac{dg}{dt'} = 1 + \frac{1}{c} \frac{d}{dt'} |\vec{r} - \vec{r}_0(t')|$$

$$\frac{d}{dt} \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = \frac{-\left[(x - x_0)\frac{dx_0}{dt} + (y - y_0)\frac{dy_0}{dt} + (z - z_0)\frac{dz_0}{dt}\right]}{\sqrt{}}$$

$$= \frac{-(\vec{r} - \vec{r}_0(t')) \cdot \vec{v}(t')}{|\vec{r} - \vec{r}_0(t')|} \equiv -\hat{n}(t') \cdot \vec{v}(t')$$

where $\hat{n}(t') \equiv \dfrac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|}$

unit vector from charges position, at retarded time t', to observer at position r, time t.

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_0(t')|} \frac{1}{1-\frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')}$$

similarly:

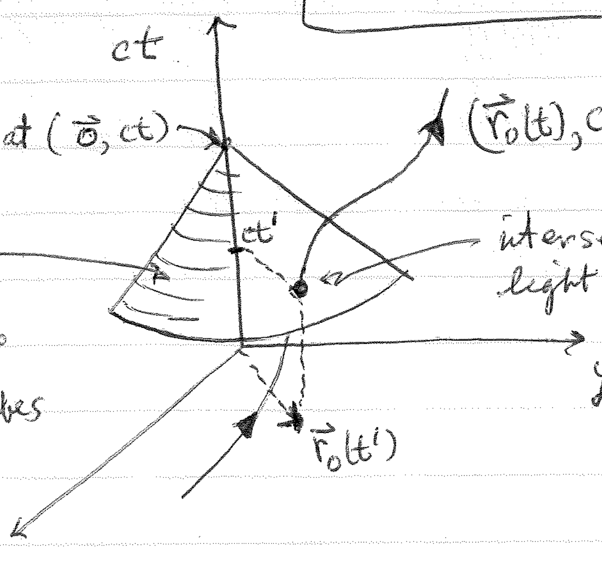
$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{q\vec{v}(t')}{|\vec{r}-\vec{r}_0(t')|} \frac{1}{1-\frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')}$$

$$t' = t - \frac{|\vec{r}-\vec{r}_0(t')|}{c}$$

Liénard-Wiechert potentials

using $\mu_0 = \frac{1}{\varepsilon_0 c^2} \Longrightarrow$ $\boxed{\vec{A}(\vec{r},t) = \frac{\vec{v}(t')}{c^2} V(\vec{r},t)}$

graphically:



"light cone"
slope of cone is 1, ie at 45°
$r = ct$ describes propagation of electromagnetic effects

intersection of trajectory with light cone, determines the retarded time t', which is responsible for the electromagnetic potentials at observation point $(\vec{0}, ct)$

what is observed at the apex of the cone, at point $(\vec{0}, ct)$ can only be caused by events which occurred somewhere on the surface of the light ~~con~~ cone.

pts r' on light cone determined by eqn $ct' = ct - r'$ as observer at $\vec{r}=0$

$\Rightarrow r' = ct - ct'$

Potential from charge moving with constant velocity

$$\vec{r}_0(t) = \vec{v}\,t$$

find retarded time $t' = t - \dfrac{|\vec{r}-\vec{r}_0(t')|}{c} \Rightarrow c(t'-t) = |\vec{r}-\vec{r}_0(t)|$

$$\longrightarrow c^2(t-t')^2 = |\vec{r}-\vec{r}_0(t')|^2 = r^2 + v^2 t'^2 - 2\vec{r}\cdot\vec{v}\,t'$$

$$c^2t^2 + c^2t'^2 - 2c^2tt' = r^2 + v^2t'^2 - 2\vec{r}\cdot\vec{v}\,t'$$

quadratic eqn in t'

$$\longrightarrow (c^2-v^2)t'^2 - 2(c^2t - \vec{r}\cdot\vec{v})t' + c^2t^2 - r^2 = 0$$

quadratic formula

$$\boxed{t' = \frac{(c^2t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 - (c^2t^2 - r^2)(c^2-v^2)}}{(c^2-v^2)}} \quad (*)$$

to get sign correct, note that when $v=0$, we want

$t' = t - \frac{r}{c}$ Apply to above:

$$t' = \frac{c^2t \pm \sqrt{c^4t^2 - c^4t^2 + c^2r^2}}{c^2}$$

$$= t \pm \frac{r}{c} \Rightarrow \text{take } (-) \text{ sign}$$

scalar potential $V(\vec{r},t) = \dfrac{q}{4\pi\varepsilon_0}\,\dfrac{1}{|r - r_0(t')|\,(1 - \frac{1}{c}\hat{n}(t')\cdot\vec{v}(t'))}$

here $\vec{v}(t') = \vec{v}$ constant

$$\underbrace{|\vec{r}-\vec{r}_0(t')|}_{=c(t-t')}\left(1 - \frac{1}{c}\hat{n}(t')\cdot\vec{v}\right) = \underbrace{|\vec{r}-\vec{v}t'|}_{=c(t-t')} - \frac{1}{c}(\vec{r}-\vec{v}t')\cdot\vec{v}$$

$$= \cancel{c(t-t')\left[1 - \frac{\vec{r}\cdot\vec{v} - v^2t'}{c^2(t-t')}\right]}$$

$$= c(t-t') - \frac{1}{c}\vec{r}\cdot\vec{v} + \frac{v^2t'}{c} = ct - \frac{\vec{r}\cdot\vec{v}}{c} - c\left(1 - \frac{v^2}{c^2}\right)t'$$

$$= \frac{1}{c}\left[\, c^2 t - \vec{r}\cdot\vec{v} - (c^2 - v^2)\, t' \,\right] \qquad \text{insert } (*) \text{ for } t'$$

$$= \frac{1}{c}\left[\, \underbrace{\cancel{c^2 t - \vec{r}\cdot\vec{v} - (c^2 t - \vec{r}\cdot\vec{v})}}_{\text{cancel}} + \sqrt{(c^2 t^2 - \vec{r}\cdot\vec{v})^2 - (c^2 t - r^2)(c^2 - v^2)} \,\right]$$

$$= \frac{1}{c}\sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 - (c^2 t^2 - r^2)(c^2 - v^2)}$$

simplify the square root

$$= \frac{1}{c}\sqrt{\cancel{c^4 t^2} + (\vec{r}\cdot\vec{v})^2 - 2c^2 t\, \vec{r}\cdot\vec{v} - \cancel{c^4 t^2} + r^2 c^2 + c^2 v^2 t^2 - r^2 v^2}$$

$$= \frac{1}{c}\sqrt{c^2(\vec{r} - \vec{v}t)^2 + (\vec{r}\cdot\vec{v})^2 - r^2 v^2}$$

$$= \sqrt{(\vec{r} - \vec{v}t)^2 + \frac{(\vec{r}\cdot\vec{v})^2}{c^2} - \frac{r^2 v^2}{c^2}}$$

$$\boxed{V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{\sqrt{(\vec{r} - \vec{v}t)^2 + \frac{(\vec{r}\cdot\vec{v})^2}{c^2} - \frac{r^2 v^2}{c^2}}}}$$

Note, for $\left(\frac{v}{c}\right)^2 \ll 1$

$$V(\vec{r},t) \approx \frac{q}{4\pi\varepsilon_0}\,\frac{1}{\sqrt{(\vec{r} - \vec{v}t)^2}} = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{|\vec{r} - \vec{r}_0(t)|^2}$$

"quasistatic" limit
looks like instantaneous
Coulomb potential

vector potential

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}\, V(\vec{r},t)$$

fields from charges moving with constant $\vec{v}$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V - \frac{\vec{v}}{c^2}\frac{\partial V}{\partial t} \qquad \text{as } \vec{A} = \frac{\vec{v}}{c^2} V$$

$$-\vec{\nabla} V = \frac{q}{4\pi\varepsilon_0} \frac{\left(\frac{1}{2}\right)}{(\quad)^{3/2}} \left[\vec{\nabla}(\vec{r}-\vec{v}t)^2 + \frac{1}{c^2}\vec{\nabla}(\vec{r}\cdot\vec{v})^2 - \frac{v^2}{c^2}\vec{\nabla} r^2 \right]$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{(\quad)^{3/2}} \left[\vec{r}-\vec{v}t + \frac{\vec{v}}{c^2}(\vec{r}\cdot\vec{v}) - \frac{v^2}{c^2}\vec{r} \right]$$

$$-\frac{\partial V}{\partial t} = \frac{q}{4\pi\varepsilon_0} \frac{\left(\frac{1}{2}\right)}{(\quad)^{3/2}} \left[\frac{\partial}{\partial t}(\vec{r}-\vec{v}t)^2 \right] = \frac{q}{4\pi\varepsilon_0} \frac{(-1)}{(\quad)^{3/2}} \left[(\vec{r}-\vec{v}t)\cdot\vec{v} \right]$$

$$\vec{E} = -\vec{\nabla} - \frac{\vec{v}}{c^2}\frac{\partial V}{\partial t} = \frac{q}{4\pi\varepsilon_0} \frac{1}{(\quad)^{3/2}} \left[\vec{r}-\vec{v}t + \frac{\vec{v}}{c^2}(\vec{r}\cdot\vec{v}) - \frac{v^2}{c^2}\vec{r} - \frac{\vec{v}}{c^2}\left(\vec{r}\cdot\vec{v} - v^2 t\right) \right]$$

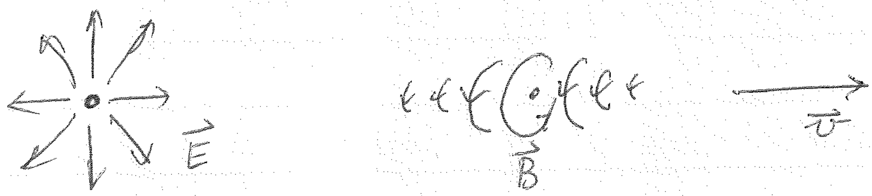
$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{v}t)\left(1 - v^2/c^2\right)}{\left((\vec{r}-\vec{v}t)^2 + \left(\frac{\vec{r}\cdot\vec{v}}{c}\right)^2 - \left(\frac{rv}{c}\right)^2 \right)^{3/2}}}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \nabla\times\left(\frac{\vec{v}}{c^2}V\right) = (\vec{\nabla}V)\times\frac{\vec{v}}{c^2} = \left(-\vec{E} - \frac{\partial \vec{A}}{\partial t}\right)\times\frac{\vec{v}}{c^2}$$

$$= -\vec{E}\times\frac{\vec{v}}{c^2} - \frac{\partial}{\partial t}\left(\frac{\vec{v}}{c^2}V\right)\times\frac{\vec{v}}{c^2} = \frac{\vec{v}}{c^2}\times\vec{E} - \frac{\partial V}{\partial t}\underbrace{\frac{\vec{v}}{c^2}\times\frac{\vec{v}}{c^2}}_{=0}$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{\vec{v}}{c^2}\times\vec{E}(\vec{r},t)}$$

Note: The factor $1 - v^2/c^2$ reminds one of special relativity!



for $v^2 \ll c^2$

$$\vec{E}(\vec{r},t) \simeq \frac{q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3}$$ instantaneous Coulomb field

$$\vec{B}(\vec{r},t) = \frac{\vec{v}}{c^2}\times\vec{E} = \frac{1}{4\pi\varepsilon_0 c^2}\, q\frac{\vec{v}\times(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3}$$

$$= \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\vec{j}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

with $\vec{j}(\vec{r}') = q\vec{v}\,\delta(\vec{r}'-\vec{v}t)$

so $\vec{B}$ looks just like Biot-Savart law, even though for moving charge, $\vec{\nabla}\cdot\vec{j} \neq 0$.

Since a charge moving with constant $\vec{v}$ can always be considered to be at rest in another frame of reference, the formulas above for $\vec{E}$ and $\vec{B}$ for charge moving with $\vec{v}$, will tell us how $\vec{E}$ and $\vec{B}$ transform under a Lorentz transformation between two frames in relative uniform motion. In fact, the Lorentz transf so defined, was first discovered as a law of electrodynamics, before the theory of special relativity was discovered.

For a pt charge in state of arbitrary motion ($\dot{\vec{v}} \neq 0$)
one finds (see text)

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}-\vec{r}_0(t')}{\left([\vec{r}-\vec{r}_0(t')]\cdot[c\hat{n}(t')-\vec{v}(t')]\right)^3}\left[[c\hat{n}(t')-\vec{v}(t')](c^2-v^2) + (\vec{r}-\vec{r}_0(t'))\times([c\hat{n}(t')-\vec{v}(t')]\times\vec{a}(t'))\right]}$$

where $t' = t - \frac{|\vec{r}-\vec{r}_0(t')|}{c}$ is retarded time

$\hat{n}(t') = \frac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$ $\qquad \vec{v} = \frac{d\vec{r}_0}{dt}$, $\vec{a} = \frac{d^2\vec{r}_0}{dt^2}$

first term $\sim \frac{1}{r^2}$ is "generalized Coulomb field" or "velocity field"

second term $\sim \frac{1}{r}$ is "acceleration field"

$$\boxed{\vec{B}(\vec{r},t) = \frac{\hat{n}(t')}{c}\times\vec{E}(\vec{r},t)}$$

$\vec{B}$ always $\perp$ $\vec{E}$, and $\vec{B} \perp \hat{n}$, vector from observer at $\vec{r}$
~~at the~~ to retarded point $\vec{r}_0(t')$.

Can use above to write down total force between
two point charges, in arbitrary states of motion (see text)

Can use to construct $\vec{S}$ and get power radiated by an
accelerating point charge - in non-relativistic limit,
one recovers Larmor's formula - see text