Given the Greens function for the wave equation in Fourier space, we do the integral to get the Greens function in real space real time

$$G(\vec{r},t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \frac{4\pi c^2}{k^2c^2-\omega^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\uparrow$ poles at $\omega = \pm ck$

In evaluating the ω integral we have to know how to treat the poles on the real axis so that $G(\vec{r},t)$ will have the desired behavior.

What we want is for $G(\vec{r},t)$ to be <u>causal</u>, ie $G(\vec{r},t)=0$ for $t<0$, so $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend only on the values of the sources at <u>earlier</u> times $t'<t$.

$$\int d^3k\, e^{i\vec{k}\cdot\vec{r}}\, \tilde{G}(\vec{k},\omega) = 2\pi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2\, e^{ikr\cos\theta}\, \tilde{G}(k,\omega)$$

$\mu \equiv \cos\theta$

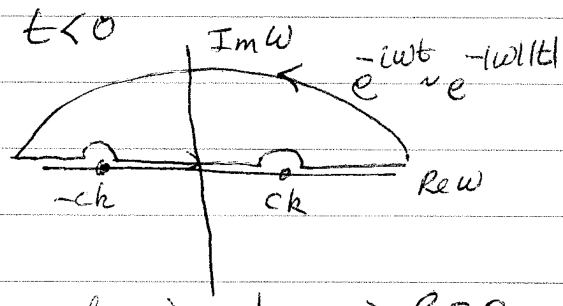
$$= 2\pi \int_{-1}^{1} d\mu \int_0^{\infty} dk\, k^2\, e^{ikr\mu}\, \tilde{G}(k,\omega)$$

$$= 4\pi \int_0^{\infty} dk\, k^2\, \frac{\sin kr}{kr}\, \tilde{G}(k,\omega)$$

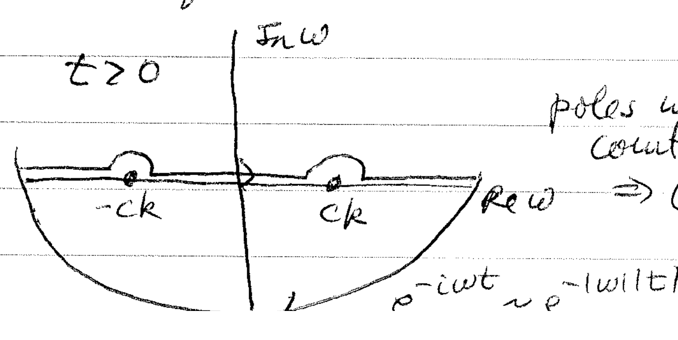
$$G(\vec{r},t) = -\frac{c^2}{\pi^2} \int_0^{\infty} dk\, k^2\, \frac{\sin kr}{kr} \int_C \frac{e^{-i\omega t}}{(\omega+ck)(\omega-ck)}\, d\omega$$

contour along real axis, but deformed to go around the poles

for $t<0$, $e^{-i\omega t}$ will decay exponentially fast for large $|\omega|$ in the <u>upper half complex</u> (UHP) ω plane $\Rightarrow$ can close contour in UHP for $t<0$. If we want $G=0$ for $t<0$, there should therefore be no poles in UHP. The contour C we want is therefore:



no poles in contour $\Rightarrow G=0$



poles within contour $\Rightarrow G\neq 0$

with this convention for the contour C we can evaluate the ω-integral using Cauchy's residue theorem

$$\int \frac{e^{-i\omega t}\, d\omega}{(\omega+ck)(\omega-ck)} = -2\pi i \left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right]$$

$$= -\frac{2\pi \sin(ckt)}{ck}$$

$$G(\vec{r}, t) = \frac{2c}{\pi r}\int_0^\infty dk \sin(kr)\sin(ckt) = \frac{c}{\pi r}\int_{-\infty}^{\infty} dk \frac{(e^{ikr}-e^{-ikr})(e^{ickt}-e^{-ickt})}{(-4)},$$

$$= -\frac{c}{2r}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{ e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k} \right.$$

each integral would give a δ-function, but for 1st two terms $\delta(r+ct)=0$ since here $t>0$ (by definition) and $r=|\vec{r}|\geq 0$ so the arguement will never vanish.

$$\boxed{G(\vec{r}, t) = \frac{c}{r}\,\delta(r-ct) = \frac{\delta(t - r/c)}{r}}$$

using $\delta(ax) = \frac{1}{a}\delta(x)$

$$G(\vec{r}t, \vec{r}'t') = \begin{cases} \dfrac{\delta\left(t-t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} & t-t'>0 \\ 0 & t-t'<0 \end{cases}$$

Green's function for wave equation in free space

$G \neq 0$ only on "light cone" that ~~connects~~ emanates from (r', t'), ie when $|\vec{r}-\vec{r}'| = c(t-t')$.

[illegible]