PHY 218 **Final Exam** **Spring 2017**

Please put a box around your final answer, and cross out any work you do not wish me to look at.

1) [25 points total]

Give a brief and to the point answer to each of the following. No lengthly calculations should be necessary.

a) [5 pts] Explain why Maxwell's correction to Ampere's Law is necessary to give charge conservation.

b) [5 pts] Why was it necessary for us to discuss the dynamical corrections to Maxwell's equations for electro- and magnetostatics in order to derive an expression for the energy stored in a *magnetostatic* magnetic field $\mathbf{B}$?

c) [5 pts] In what range of frequencies is EM wave propagation in a conducting metal most different from in a non-conducting dielectric? Explain the reason for the difference.

d) [5 pts] In what situations does Snell's law, relating angle of transmission to angle of incidence at an interface, *not* apply?

e) [5 pts] A swimmer, swimming underwater in an outdoor pool, tilts her head to look upwards towards the water surface. But instead of seeing the sky, or the people in lounge chairs by the side of the pool, she sees the swimmers swimming underwater in front of her! Explain how this can happen.

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2) [20 points total]

The electric and magnetic fields of a plane electromagnetic wave traveling along the $\hat{\mathbf{z}}$ axis, in a dissipative medium, can be written as:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 \mathrm{e}^{-k_2 z} \mathrm{e}^{i(k_1 z - \omega t)}\right], \quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[\frac{|k|}{\omega}(\hat{\mathbf{z}} \times \mathbf{E}_0) \mathrm{e}^{-k_2 z} \mathrm{e}^{i(k_1 z - \omega t + \phi)}\right]$$

where k_1 and k_2 are the real and imaginary parts of the wave vector $k = k_i + ik_2$, $|k| = \sqrt{k_1^2 + k_2^2}$, and $\tan(\phi) = k_2/k_1$.

For a *linearly* polarized plane wave, where the amplitude $\mathbf{E}_0$ is a real vector, $\mathbf{E}$ and $\mathbf{B}$ are orthogonal, i.e. $\mathbf{E} \cdot \mathbf{B} = 0$. However, for a *circularly* polarized wave, with complex $\mathbf{E}_0 = E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$, this is no longer necessarily true.

a) [10 pts] Compute the value of $\mathbf{E} \cdot \mathbf{B}$ for a circularly polarized wave.

b) [5 pts] What is the angle between $\mathbf{E}$ and $\mathbf{B}$? How does it vary as a function of position and time?

c) [5 pts] Under what physical conditions (i.e., for waves in what type of material) will $\mathbf{E} \cdot \mathbf{B} = 0$ for the circularly polarized wave?

3) [30 points total]

For a time dependent charge distribution with an oscillating electric dipole moment $\mathbf{p}(t) = \text{Re}[\mathbf{p}_\omega e^{-i\omega t}]$, the radiated fields (i.e. in the radiation zone) in the electric dipole approximation are given by,

$$\mathbf{E}(\mathbf{r},t) = \text{Re}\left[\frac{k^2}{4\pi\epsilon_0}\frac{e^{i(kr-\omega t)}}{r}\hat{\mathbf{r}}\times(\mathbf{p}_\omega\times\hat{\mathbf{r}})\right], \quad \mathbf{B}(\mathbf{r},t) = \text{Re}\left[\frac{-c\mu_0 k^2}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\mathbf{p}_\omega\times\hat{\mathbf{r}}\right],$$

where $k = \omega/c$.

Consider a point charge q moving in a circular orbit of radius R, centered about the origin in the xy plane. The charge is orbiting counterclockwise with an angular velocity ω.

a) [15 pts] Compute the radiated electric and magnetic fields in the electric dipole approximation, expressing your answer in spherical coordinates, i.e. $\mathbf{E}(r,\theta,\varphi,t)$ and $\mathbf{B}(r,\theta,\varphi,t)$. Make sure your answers are real valued functions of position and time!

b) [10 pts] What is the polarization of the outgoing radiation when $\theta = 0$? when $\theta = \pi/2$?

c) [5 pts] What is the total radiated energy per one orbit of the charge?

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4) [25 pts]

Consider the 2nd rank symmetric 4-tensor, defined by,

$$T_{\mu\nu} = \frac{1}{\mu_0}\left[F_{\mu\lambda}F_{\lambda\nu} - \frac{1}{4}\delta_{\mu\nu}F_{\lambda\sigma}F_{\sigma\lambda}\right]$$

where $F_{\mu\lambda}$ is the field strength tensor (remember the summation convention!).

a) [10 pts] By explicit calculation show that,

$$T_{\mu\nu} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} & -icp_x \\ T_{yx} & T_{yy} & T_{yz} & -icp_y \\ T_{zx} & T_{zy} & T_{zz} & -icp_z \\ -icp_x & -icp_y & -icp_z & u \end{pmatrix}$$

where u is the electromagnetic field energy density, $\mathbf{p} = (p_x, p_y, p_z)$ is the electromagnetic field momentum density, and T_{ij} are the components of the Maxwell stress tensor. $T_{\mu\nu}$ is called the covariant stress tensor.

b) [8 pts] Using the definition of $T_{\mu\nu}$ in terms of $F_{\mu\nu}$ above, and using the covariant form of the inhomogeneous Maxwell's equations $\partial F_{\mu\nu}/\partial x_\nu = \mu_0 j_\mu$, with j_μ the 4-current, one can show (but you don't need to derive it) that,

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = F_{\mu\nu}j_\nu.$$

Show that the above result is a covariant way to write the laws of local conservation of momentum and energy.

c) [7 pts] Consider the total energy and total momentum carried by the electromagnetic fields,

$$U = \int d^3r\, u(\mathbf{r}), \qquad \mathbf{P} = \int d^3r\, \mathbf{p}(\mathbf{r}),$$

where the integrals go over all of space. Argue from part (a) why iU/c and $\mathbf{P}$ do not in general give the time and space components of a covariant energy-momentum 4-vector, as one might naively expect. A good clear argument, rather than a detailed calculation, is all you need to give.

(This last part (c) is meant to illustrate some of the difficulties associated with defining a total energy-momentum 4-vector for electromagnetic fields. For the solution to these difficulties, see Jackson, *Classical Electrodynamics*, 1st edition, sections 17.5 and 17.6. These sections are reasonably readable.)

Some helpful formulae for Problem 4

The Lorentz transformation matrix from a frame $\mathcal{K}$ to a frame $\mathcal{K}'$ that moves with velocity $v\hat{\mathbf{x}}$ with respect to $\mathcal{K}$ is:

$$a_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

The field strength tensor $F_{\mu\nu}$ is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & -B_x & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{pmatrix}$$

The energy density u, energy current $\mathbf{S}$, momentum density $\mathbf{p}$, and Maxwell stress tensor T_{ij} for electromagnetic fields are:

$$u = \frac{1}{2}\left(\epsilon_0|\mathbf{E}|^2 + \frac{1}{\mu_0}|\mathbf{B}|^2\right) \qquad \mathbf{p} = \frac{1}{c^2}\mathbf{S} = \frac{1}{\mu_0 c^2}\mathbf{E}\times\mathbf{B}$$

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$