Solutions to the wave equation $\Box^2 f = 0$.

plane waves: if $g(\phi)$ is any function of single variable ϕ, then

$f(\vec{r},t) = g(\vec{k}\cdot\vec{r} - \omega t)$ solves wave equation, for any vector $\vec{k}$, and for $\omega^2 = v^2 k^2$

proof:

$$\Box^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial g}{\partial \phi} k_x \quad \text{chain rule}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial \phi}\frac{\partial \phi}{\partial x}\right) = \frac{\partial g}{\partial \phi}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 g}{\partial \phi^2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{\partial^2 g}{\partial \phi^2} k_x^2$$

similarly $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 g}{\partial \phi^2} k_y^2$, $\dfrac{\partial^2 g}{\partial z^2} = \dfrac{\partial^2 g}{\partial \phi^2} k_z^2$, $\dfrac{\partial^2 g}{\partial t^2} = \dfrac{\partial^2 g}{\partial \phi^2} \omega^2$

$$\Box^2 f = \left(k^2 - \frac{\omega^2}{v^2}\right)\frac{\partial^2 g}{\partial \phi^2} = 0 \quad \text{if} \quad \omega^2 = v^2 k^2.$$ for any function $g(\phi)$

~~ω is frequency of the wave, i.e. ν = ω/2π, T = 1/ν is period~~

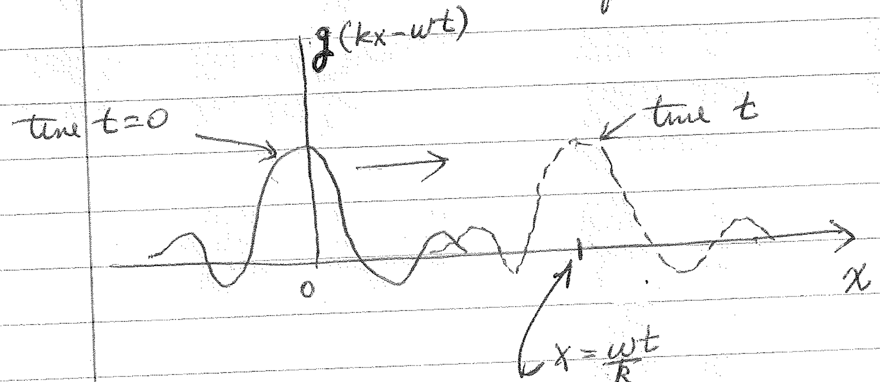
~~$f(\vec{r}, t+T) = f(\vec{k}\cdot\vec{r} - \omega(t+T))$~~
~~$= f(\vec{k}\cdot\vec{r} - \omega t - \omega 2\pi/\omega)$~~

plane wave because $f(\vec{r},t)$ is constant on all planes $\perp$ to $\vec{k}$

i.e. if $\Delta\vec{r}$ is $\perp$ to $\vec{k}$, then

$$f(\vec{r}+\Delta\vec{r},t) = g(\vec{k}\cdot\vec{r} + \underbrace{\vec{k}\cdot\Delta\vec{r}}_{=0} - \omega t) = g(\vec{k}\cdot\vec{r} - \omega t) = f(\vec{r},t)$$

v is ~~velocit~~ speed of wave, suppose $\vec{k} = k\hat{x}$



$g(\phi)$ is called the envelope of the wave

curve shifted to right a distance $x = \frac{\omega}{k}t$ in time t.
$\Rightarrow$ curve moves with velocity $\vec{v} = \frac{\omega}{k}\hat{x}$

spherical waves

solution to wave eqn that depends only on radial coord r, ie $f(r,t)$ ← f const on spheres of radius r

$$\left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] f(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

try solution of form $\boxed{f(r,t) = \frac{g(kr-\omega t)}{r}}$

call $\phi = kr - \omega t$

$$r^2\left(\frac{\partial f}{\partial r}\right) = r^2\left(\frac{1}{r}\frac{dg}{d\phi}k - \frac{g}{r^2}\right) = r\frac{dg}{d\phi}k - g$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\left[\frac{dg}{d\phi}k + r\frac{d^2g}{d\phi^2}k^2 - \frac{dg}{d\phi}k\right] = \frac{d^2g}{d\phi^2}\frac{k^2}{r}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{r}\frac{d^2 g}{d\phi^2}\omega^2$$

so $\frac{g}{r}(kr-\omega t)$ solves wave eqn provided $\frac{d^2g}{d\phi^2}\frac{k^2}{r} = \frac{1}{v^2}\frac{d^2g}{d\phi^2}\frac{\omega^2}{r}$

ie $\omega^2 = v^2 k^2$

Note: if $f_1(\vec{r},t)$ and $f_2(\vec{r},t)$ are solutions to wave eqn, then so is $f_1 + f_2$, as $\Box^2$ is a <u>linear</u> operator

<u>Back to plane waves</u>

A particular solution: sinusoidal wave

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

period of wave is $T = \frac{2\pi}{\omega}$: $f(\vec{r}, t+T) = f(t)$

frequency of wave is $\nu = \omega/2\pi = 1/T$

<u>angular freq</u> is ω

wavelength is $\lambda = \frac{2\pi}{|\vec{k}|}$: $f(\vec{r} + \lambda\hat{k}, t) = f(\vec{r},t)$

<u>wave vector</u> is $\vec{k}$ <u>wave number</u> is $k = |\vec{k}|$

<u>amplitude</u> is A

<u>phase constant</u> is δ

phase velocity is $\vec{v} = \frac{\omega}{|\vec{k}|}\hat{k}$ travels in direction $\hat{k}$

<u>ex:</u> if $\vec{k} = k\hat{x}$ then wave travels in $+\hat{x}$ directi

if $\vec{k} = -k\hat{x}$ then $A\cos(-kx - \omega t + \delta)$ travels in $-\hat{x}$ direction

<u>Complex notation</u> $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(\vec{r},t) = \mathrm{Re}\left[A e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta)}\right]$$

usually we will not write the "Re" but leave it implied.

$$f(\vec{r},t) = \mathrm{Re}\left[\underbrace{A e^{i\delta}}_{\text{complex amplitude}} e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]$$

Sinusoidal waves particularly important, since we can expand any solution of wave eqn in terms of them
$\Rightarrow$ theory of Fourier Transforms

Fourier Series: Any function $f(x)$ defined on $[-\frac{L}{2}, \frac{L}{2}]$ can be expressed as

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right) \right]$$

where $$A_n = \frac{2}{L} \int_{-\frac{L}{2}}^{L/2} dx\, f(x) \cos\left(\frac{2\pi n x}{L}\right)$$

$$B_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) \sin\left(\frac{2\pi n x}{L}\right)$$

rewrite in terms of complex exponential.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ \left(\frac{A_n}{2} + \frac{B_n}{2i}\right) e^{i\frac{2\pi n x}{L}} + \left(\frac{A_n}{2} - \frac{B_n}{2i}\right) e^{-i\frac{2\pi n x}{L}} \right\}$$

where $$\frac{A_n}{2} \pm \frac{B_n}{2i} = \frac{A_n}{2} \mp \frac{iB_n}{2} = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, \frac{1}{2} \left\{ \cos\frac{2\pi n x}{L} \mp i \sin\frac{2\pi n x}{L} \right\}$$

$$= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{\mp i \frac{2\pi n x}{L}}$$

$n = 1, 2, \ldots$

Define $f_n \equiv \frac{A_n}{2} + \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi n x}{L}}$

$f_{-n} \equiv \frac{A_n}{2} - \frac{B_n}{2i} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi(-n)x}{L}}$

$f_0 \equiv \frac{A_0}{2} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)$

$$f(x) = f_0 + \sum_{n=1}^{\infty}\left\{ f_n e^{i\frac{2\pi n x}{L}} + f_{-n} e^{i 2\pi(-n)x/L} \right\}$$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i 2\pi n x/L}, \quad f_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i\frac{2\pi n x}{L}}}$$

Fourier series in Complex form

Now let $L \to \infty$

Define $k_n = \frac{2\pi n}{L}$, $k_{n+1} - k_n = \Delta k = \frac{2\pi}{L}$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} L\, f_n\, e^{i k_n x}$$

Define $\tilde{f}(k_n) \equiv \frac{L}{2\pi} f_n = \frac{1}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x)\, e^{-i k_n x}$

$$f(x) = \sum_{n=-\infty}^{\infty} \Delta k\, \tilde{f}(k_n)\, e^{i k_n x}$$

as $L \to \infty$, $\Delta k \to 0$, $\sum \Delta k \to \int dk$

$$\boxed{f(x) = \int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{ikx}, \quad \tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}}$$

Fourier transform

$\tilde{f}(k)$ is the Fourier transform of $f(x)$.

$$f = f^*$$

If $f(x)$ is a real function, then

$$\tilde{f}(-k) = \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, f(x)\, e^{+ikx} = \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, f^*(x) \left(e^{-ikx}\right)^*$$

$$= \left(\int_{-\infty}^{\infty} \frac{dx}{2\pi}\, f(x)\, e^{-ikx}\right)^* = \tilde{f}^*(+k)$$

$$\boxed{\tilde{f}(-k) = \tilde{f}^*(k)}$$

For a function of 3-dim space,

$$f(x,y,z) = \int_{-\infty}^{\infty} dk_x\, dk_y\, dk_z\, \tilde{f}(k_x, k_y, k_z)\, e^{ik_x x}\, e^{ik_y y}\, e^{ik_z z}$$

or

$$\boxed{f(\vec{r}) = \int_{-\infty}^{\infty} d^3k\, \tilde{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}}$$

$$\boxed{\text{where} \quad \tilde{f}(\vec{k}) = \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3}\, f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}}$$

For function of $\vec{r}$ and t

$$f(\vec{r}, t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \tilde{f}(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where

$$\tilde{f}(\vec{k}, \omega) = \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, f(\vec{r}, t)\, e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$$

Fourier transfoms are of enormous help in solving partial differential equations.

General solution to wave equation

$\Box^2 f(\vec{r},t) = 0$ subst in F.T.

$$\Box^2 \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega) \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right) = \vec{\nabla}\cdot\begin{pmatrix} \frac{\partial}{\partial x} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial y} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial z} e^{i(k_x x + k_y y + k_z z)} \end{pmatrix}$$

$$= \vec{\nabla}\cdot\begin{pmatrix} ik_x \\ ik_y \\ ik_z \end{pmatrix} e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\right)$$

product rule:

$$= i\vec{k}\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\Rightarrow \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \left(-k^2 + \frac{\omega^2}{v^2}\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

So wave equ is

$$\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega)\left(-k^2 + \frac{\omega^2}{v^2}\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0$$

$$\Rightarrow \tilde{f}(\vec{k},\omega)\left(-k^2 + \frac{\omega^2}{v^2}\right) = 0$$

: if a function $=0$, then its F.T. $=0$ (all Fourier coefficients $=0$)

either

$\Rightarrow \tilde{f}(\vec{k},\omega) = 0$, ie there is no such $(\vec{k},\omega)$ component

or $k^2 v^2 = \omega^2$. $\therefore$ for each $\vec{k}$, only $\omega = v|\vec{k}|$ is present

So most general solution is

$$f(\vec{r},t) = \int_{\infty}^{\infty} d^3k \, \tilde{f}(\vec{k}) \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{where } \omega^2 = v^2 k^2$$

[no longer integrate over ω since for each $\vec{k}$, ω fixed by]
and $\tilde{f}(\vec{k})$ is anything

General solution is ~~sum~~ linear combination of sinusoidal waves.

In most problems therefore, it will be enough to determine how plane sinusoidal waves with wavevector $\vec{k}$ behave. The general solution can then always be represented as a linear combination of these plain sinusoidal waves.

Alternatively, if $f(r, t=0) = f_0(\vec{r})$ a given function, then this will evolve in time according to

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \; \tilde{f}_0(k) \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \omega^2 = v^2 k^2$$

$$\text{where} \quad \tilde{f}_0(\vec{k}) = \int \frac{d^3r}{(2\pi)^3} f_0(\vec{r}) \, e^{-i\vec{k}\cdot\vec{r}}$$

↑ Fourier transform of f at $t=0$