Conductors

conduction electrons are free $\Rightarrow$ give $\vec{J}_f$ and ρ_f

$$m\ddot{\vec{r}} = -e\vec{E}(t) - \frac{m}{\tau}\dot{\vec{r}} \qquad \tau \text{ is "collision time"}$$

$$\ddot{\vec{r}} + \frac{\dot{\vec{r}}}{\tau} = -\frac{e}{m}\vec{E}$$

just like polarizable atom except $\omega_0 = 0$ – no restoring force

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}$$

$$\Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t}$$

$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\vec{r}_\omega = -\frac{e}{m}\vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m}\frac{1}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega$$

$$= -\frac{e\tau}{m\omega}\frac{i}{1-i\omega\tau}\vec{E}_\omega$$

current flowing is $\vec{J}_f = -eN\vec{v}$

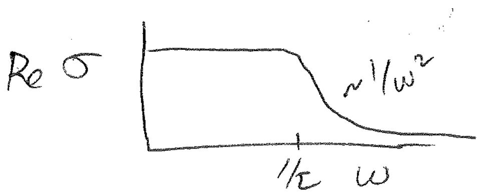
$$= -eN\dot{\vec{r}} \qquad N = \text{density conduction electrons}$$

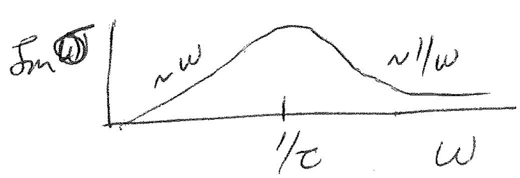
$$\vec{J}_f = \vec{J}_\omega e^{-i\omega t}, \quad \vec{J}_\omega = -eN(-i\omega)\vec{r}_\omega$$

$$= \frac{Ne^2\tau}{m}\frac{1}{1-i\omega\tau}\vec{E}_\omega$$

Define freq dependent conductivity

$$\boxed{\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega} \Rightarrow \boxed{\sigma(\omega) = \frac{Ne^2\tau}{m}\frac{1}{1-i\omega\tau}}$$





$$\mathrm{Re}\,\sigma = \frac{\sigma_0}{1+\omega^2\tau^2}$$

$$\mathrm{Im}\,\sigma = \frac{\sigma_0\omega\tau}{1+\omega^2\tau^2}$$

charge density obtained by charge conservation

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{J}_f$$

For a plane wave $\quad \vec{J}_f = \vec{J}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$\rho_f = \rho_\omega e^{i(k\cdot r-\omega t)}$$

$$-i\omega\rho_\omega = -i\vec{k}\cdot\vec{J}_\omega \quad \Rightarrow \quad \boxed{\rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega}}$$

Maxwell's Equ $\qquad \vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ etc.

1) $\vec{\nabla}\cdot\vec{D} = \rho_{free}$ $\qquad$ 2) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\qquad$ 4) $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}_{free}$

assume $\quad \vec{H} = \frac{\vec{B}}{\mu}$, μ constant

$\vec{D}_\omega = \varepsilon_b(\omega)\vec{E}_\omega$ $\qquad$ $\varepsilon_b(\omega)$ dielectric response from bound electrons

$\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega$ $\qquad$ $\sigma(\omega)$ conductivity due to free electrons

$$\rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega} = \frac{\sigma(\omega)}{\omega}\vec{k}\cdot\vec{E}_\omega$$

1) $\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow i\vec{k}\cdot\vec{D}_\omega = \rho_\omega$

$$\Rightarrow i\vec{k}\cdot\varepsilon_b(\omega)\vec{E}_\omega = \frac{\sigma(\omega)}{\omega}\vec{k}\cdot\vec{E}_\omega$$

$$i\vec{k}\cdot\vec{E}_\omega\left[\varepsilon_b(\omega) + i\frac{\sigma(\omega)}{\omega}\right] = 0$$

2) $i\vec{k}\cdot\mu\vec{H}_\omega = 0$

3) $i\vec{k}\times\vec{E}_\omega = i\omega\vec{B}_\omega = i\omega\mu H_\omega$

4) $i\vec{k}\times H_\omega = -i\omega\varepsilon_b(\omega)\vec{E}_\omega + \sigma(\omega)\vec{E}_\omega$

$$= -i\omega\left[\varepsilon_b(\omega) + i\frac{\sigma(\omega)}{\omega}\right]\vec{E}_\omega$$

Equation have exactly the same form as for waves in a dielectric provided we use

$$\boxed{\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{i\sigma(\omega)}{\omega}}$$

<u>transverse waves</u>
$\vec{E}\perp\vec{k}$

and replace μ_0 by μ.

dispersion relation for [transverse] waves is given by

$$\boxed{k^2 = \omega^2\mu\varepsilon = \frac{\omega^2}{c^2}\frac{\mu}{\mu_0}\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

with $\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{i\sigma(\omega)}{\omega}$

[Note: for transverse mode, $\vec{k} \perp \vec{E}_\omega$, so $\vec{k} \perp \vec{j}_\omega = \sigma(\omega)\vec{E}_\omega$

$\Rightarrow \rho_\omega = \frac{\vec{k}\cdot\vec{j}_\omega}{\omega} = 0$ no charge density oscillation!]

The main difference between wave propagation in dielectrics & conductors has to do with the contribution that the $\frac{i\sigma(\omega)}{\omega}$ term makes to the real & imaginary parts of $\varepsilon(\omega)$

For our simple model (Drude model)

$\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau}$ where $\sigma_0 = \sigma(0) = \frac{Ne^2\tau}{m}$ is d.c. conductivity

① Low frequencies $\omega \ll 1/\tau$, $\omega \ll \omega_0$

ω_0 is resonant freq of ε_b

$\varepsilon_b(\omega) \approx \varepsilon_b(0)$ real

$\sigma(\omega) \approx \sigma_0$ real $\sim \tau$

$$\boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} \approx \frac{\varepsilon_b(0)}{\varepsilon_0} + \frac{i\sigma_0}{\varepsilon_0\omega}}$$

gives large imaginary part to $\varepsilon(\omega)$

grows as $\frac{1}{\omega}$ as $\omega \to 0$

$\Rightarrow$ strong dissipation

② High frequencies $\omega \gg 1/\tau$, $\omega \gg \omega_0$

$\frac{\varepsilon_b(\omega)}{\varepsilon_0} \approx 1$

$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{iNe^2\tau}{m\omega\tau} = \frac{iNe^2}{m\omega}$ imaginary indep of τ

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 + \frac{i\sigma}{\varepsilon_0\omega} \approx 1 - \frac{Ne^2}{\varepsilon_0 m\omega^2} = \boxed{1 - \frac{\omega_p^2}{\omega^2} = \frac{\varepsilon(\omega)}{\varepsilon_0}}$$

where $\omega_p = \sqrt{Ne^2/\varepsilon_0 m}$ is "plasma freq" of conduction electrons

(1) Behavior at low freq

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = \frac{\varepsilon_b(0)}{\varepsilon_0} + \frac{i\sigma_0}{\varepsilon_0\omega} = \frac{\varepsilon_b(0)}{\varepsilon_0}\left(1 + \frac{i\sigma_0}{\varepsilon_b(0)\omega}\right)$$

Dissipation is due to $\varepsilon_2 = \mathrm{Im}\,\varepsilon$

Dissipation dominate when $\varepsilon_2 \gg \varepsilon_1 = \mathrm{Re}\,\varepsilon$

ie when $\dfrac{\sigma_0}{\varepsilon_b(0)\omega} \gg 1$ — this regime is called a "good" conductor – conduction electrons playing dominant role — waves strongly attenuated

opposite limit: $\dfrac{\sigma_0}{\varepsilon_b(0)\omega} \ll 1$ — this regime is called a "poor" conductor – waves propagate transparently – little relative absorption of energy from conduction electrons

One always gets into the "good" conductor limit as ω decreases. For good conductor,

$$k \simeq \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0}} \simeq \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}}\sqrt{i\,\frac{\varepsilon_2}{\varepsilon_0}} = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}}\sqrt{\frac{\sigma_0}{\varepsilon_0\omega}}\sqrt{i}$$

$$k = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}\frac{\sigma_0}{\varepsilon_0\omega}}\left(\frac{1+i}{\sqrt{2}}\right) \Rightarrow k_1 = k_2$$

real and imaginary parts of k are equal

$\frac{1}{c} = \sqrt{\mu_0 \epsilon_0}$

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{\mu}{\epsilon_0 \mu_0}\frac{\sigma_0}{2\omega}} = \sqrt{\frac{\mu \sigma_0}{2}\omega} \sim \sqrt{\omega}$$

waves have form $\vec{E} = E_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$

decay length of amplitude is

$$1/k_2 = \sqrt{\frac{2}{\mu \sigma_0 \omega}} = \delta$$ called the "skin depth"

Faraday cage

δ is distance wave penetrates into conductor

$\delta \sim 1/\sqrt{\omega}$ gets larger as ω decreases

$$\vec{H} = \vec{H}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t + \phi)} \qquad \left|\frac{H_\omega}{E_\omega}\right| = \frac{|k|}{\omega\mu}$$

phase shift between $\vec{H}$ and $\vec{E}$ is ϕ

given by $\tan\phi = k_2/k_1 \approx 1$

$$\Rightarrow \phi \approx 45^\circ$$

Amplitude ratio $$\frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{|k|}{\omega\mu} = \frac{\sqrt{2}\,k_1}{\omega\mu} = \frac{\sqrt{2}}{\omega\mu}\sqrt{\frac{\mu\sigma_0\omega}{2}}$$

$$= \sqrt{\frac{\sigma_0}{\omega\mu}}$$ increases as $\frac{1}{\sqrt{\omega}}$ as $\omega \to 0$

$\Rightarrow$ as $\omega \to 0$, most of energy of wave is carried by the magnetic field part of the wave.

② Behavior at high freq

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \left(\frac{\omega_p}{\omega}\right)^2 \qquad \omega_p^2 = \frac{Ne^2}{m\varepsilon_0} \qquad \text{plasma freq}$$

$\varepsilon(\omega)$ is real $\quad (\varepsilon_2 \ll \varepsilon_1)$

1) If $\omega > \omega_p$, then $\varepsilon > 0$

$\Rightarrow$ transparent propagation $\qquad k_1 = \frac{\omega}{c}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

k is pure real $\qquad k_2 \gtrsim 0$

2) If $\omega < \omega_p$, then $\varepsilon < 0$

$\Rightarrow$ total reflection $\qquad k_1 = 0$

$$k_2 = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}\left(\frac{\omega_p^2}{\omega^2} - 1\right)}$$

k is pure imaginary

plasma freq ω_p gives cross over between reflection + transparent propagation.

$\tau \sim 10^{-14}$ sec for typical metal

$\omega_p \simeq 10^{16}\ \text{sec}^{-1}$ for most metals

$$\lambda_p \equiv \frac{2\pi c}{\omega_p} \sim 3\times10^3\,\text{Å} \qquad (\text{visible is } \lambda \sim 5\times10^3\,\text{Å})$$

Example: The ionosphere is a layer of charged gas surrounding the earth. In may respects the charged gas behaves like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio propagates through ionosphere + escapes into space

Explains why you can pick up AM stations from far away - they are reflected back by ionosphere - but you only pick up local FM stations - they do not get reflected by ionosphere.