$\Rightarrow R_\perp = R_\parallel = 1$ ~~when material b~~

this confirms that material b is totally reflecting in this region of frequency

When medium b is <u>transparent</u>, ie ε_b is real and $\varepsilon_b > 0$ we have

$$k_{Iz} = \omega\sqrt{\mu_a \varepsilon_a}\cos\theta_I = \frac{\omega}{c} n_a \cos\theta_I$$
$$k_{Tz} = \omega\sqrt{\mu_b \varepsilon_b}\cos\theta_T = \frac{\omega}{c} n_b \cos\theta_T$$

and Snell's Law applies, so $n_a \sin\theta_I = n_b \sin\theta_T \Rightarrow \frac{n_b}{n_a} = \frac{\sin\theta_I}{\sin\theta_T}$

we can now write $R_\perp$ and $R_\parallel$ as functions of θ_I

For simplicity take $\mu_a = \mu_b = \mu_0$

① $$R_\perp = \left(\frac{n_a\cos\theta_I - n_b\cos\theta_T}{n_a\cos\theta_I + n_b\cos\theta_T}\right)^2 = \left(\frac{\cos\theta_I - \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T}\right)^2$$

$$= \left(\frac{\sin\theta_T\cos\theta_I - \sin\theta_I\cos\theta_T}{\sin\theta_T\cos\theta_I + \sin\theta_I\cos\theta_T}\right)^2 = \left(\frac{\sin(\theta_I - \theta_T)}{\sin(\theta_I + \theta_T)}\right)^2$$

~~unmarino~~

for $\theta_I = 0$, ie normal incidence, $\theta_I = \theta_T = 0$

$$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$$ if $n_a = n_b$, no reflection!

(2) $$R_{\parallel} = \left(\frac{\varepsilon_b n_a \cos\theta_I - \varepsilon_a n_b \cos\theta_T}{\varepsilon_b n_a \cos\theta_I + \varepsilon_a n_b \cos\theta_T}\right)^2$$

use $\sqrt{\varepsilon_b \mu_0} = \frac{n_b}{c}$

$\Rightarrow \varepsilon_b = \frac{n_b^2}{c^2 \mu_0} = n_b^2 \varepsilon_0$

$\varepsilon_a = n_a^2 \varepsilon_0$

$$= \left(\frac{n_b \cos\theta_I - n_a \cos\theta_T}{n_b \cos\theta_I + n_a \cos\theta_T}\right)^2$$

$$= \left(\frac{\cos\theta_I - \left(\frac{\sin\theta_T}{\sin\theta_I}\right)\cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_T}{\sin\theta_I}\right)\cos\theta_T}\right)^2 = \left(\frac{\sin\theta_I\cos\theta_I - \sin\theta_T\cos\theta_T}{\sin\theta_I\cos\theta_I + \sin\theta_T\cos\theta_T}\right)^2$$

$$R_{\parallel} = \left(\frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)}\right)^2 \Leftarrow \text{after some algebra}$$

for $\underline{\theta_I = 0}$ $\Rightarrow \theta_T = 0$

$$R_{\parallel} = \left(\frac{\varepsilon_b n_a - \varepsilon_a n_b}{\varepsilon_b n_a + \varepsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2$$

same as for $R_\perp$! But when $\theta_I = 0$ there is no distinction between the two case $\perp$ and $\parallel$, so this is to be expected.

When $\theta_I + \theta_T = \pi/2$, then

$\tan(\theta_I + \theta_T) \to \infty$ and $R_{\parallel} = 0$.

This occurs at an angle of incidence $\theta_I \equiv \theta_B$ "Brewster's angle"

θ_B determined by $\underset{\underset{\theta_I}{\parallel}}{n_a \sin\theta_B} = n_b \sin(\underbrace{\tfrac{\pi}{2} - \theta_B}_{=\theta_T}) = n_b \cos\theta_B$

$$\Rightarrow \boxed{\tan\theta_B = \frac{n_b}{n_a}}$$

For a wave incident at θ_B, the reflected wave will always have $\vec{E}_R \perp$ plane of incidence, no matter what orientation of incoming $\vec{E}_I$, since $R_{\parallel} = 0$. ~~That is only $R_{\perp} \neq 0$, so reflected wave can only have $\vec{E} \perp$ plane of incidence.~~

If incoming wave has component of $\vec{E}_I \parallel$ to plane of incidence, this component gets purely transmitted since $R_{\parallel} = 0$. Only the component of $\vec{E}_I \perp$ to plane of incidence can get reflected, since $R_{\perp} \neq 0$. $\Rightarrow$ reflected wave is <u>polarized</u> with $\vec{E}_R \perp$ to plane of incidence.

Generally, for all θ_I close to θ_B, $R_{\parallel} \ll R_{\perp}$, and the reflected wave is strongly polarized with $\vec{E}_R$ mostly $\perp$ to plane of incidence.

This is therefore one method to create a polarized light wave.

Additional notes on Reflection & Transmission Coefficients

For a transparent medium, the energy current can be written as (see text 9.3.1)

$$\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) = \vec{E}\times\vec{H} \qquad (\text{in a vacuum } \mu = \mu_0)$$

For a plane wave $\vec{E}(\vec{r},t) = \vec{E}_\omega \cos(\vec{k}\cdot\vec{r}-\omega t)$ $\quad \vec{E}_\omega \perp \vec{k}$

From lecture 13 we have

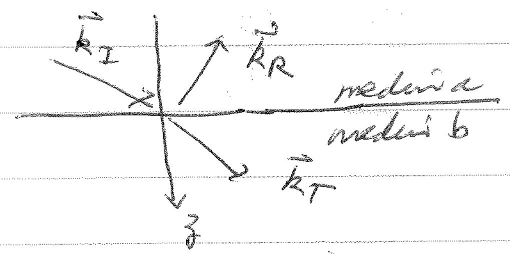
$$\vec{H}(\vec{r},t) = \frac{\vec{B}(\vec{r},t)}{\mu} = (\hat{k}\times\vec{E}_\omega)\frac{|k|}{\omega\mu}\cos(\vec{k}\cdot\vec{r}-\omega t)$$

(since the medium is transparent, k is real and $k_2 = 0$, $\delta = \arctan\frac{k_2}{k_1} = 0$)

$$\Rightarrow \quad \vec{S} = \vec{E}\times\vec{H} = \frac{|k|}{\omega\mu}\underbrace{\vec{E}_\omega\times(\hat{k}\times\vec{E}_\omega)}_{=|\vec{E}_\omega|^2\hat{k} \text{ since } \hat{k}\perp\vec{E}_\omega}\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

so

$$\langle\vec{S}\rangle = \frac{|k|}{2\omega\mu}|\vec{E}_\omega|^2\hat{k} \qquad \text{as } \langle\cos^2(\vec{k}\cdot\vec{r}-\omega t)\rangle = 1/2$$



The energy flux of the incident wave going into medium b is

$$\langle\vec{S}_I\rangle\cdot\hat{z} = \frac{|k_I|}{2\omega\mu_a}|\vec{E}_{I\omega}|^2(\hat{k}_I\cdot\hat{z}) = \frac{|k_I|}{2\omega\mu_a}|\vec{E}_{I\omega}|^2\cos\theta_I$$

The energy flux of the reflected wave is

use $\theta_I = \theta_R$

$$\langle\vec{S}_R\rangle\cdot\hat{z} = \frac{|k_R|}{2\omega\mu_a}|\vec{E}_{R\omega}|^2(\hat{k}_R\cdot\hat{z}) = \frac{|k_R|}{2\omega\mu_a}|\vec{E}_{R\omega}|^2(-\cos\theta_I)$$

(−) because reflected back

The energy flux of the transmitted wave is

$$\langle\vec{S}_T\rangle\cdot\hat{z} = \frac{|k_T|}{2\omega\mu_b}|\vec{E}_{T\omega}|^2(\hat{k}_T\cdot\hat{z}) = \frac{|k_T|}{2\omega\mu_b}|\vec{E}_{T\omega}|^2\cos\theta_T$$

Energy in = Energy out

$$\Rightarrow \langle \vec{S}_I \rangle \cdot \hat{z} + \langle \vec{S}_R \rangle \cdot \hat{z} = \langle \vec{S}_T \rangle \cdot \hat{z}$$

or $$\langle \vec{S}_I \rangle \cdot \hat{z} = \langle \vec{S}_T \rangle \cdot \hat{z} - \langle \vec{S}_R \rangle \cdot \hat{z}$$

this term is >0 this term is <0

$$|\langle \vec{S}_I \rangle \cdot \hat{z}| = |\langle \vec{S}_T \rangle \cdot \hat{z}| + |\langle \vec{S}_R \rangle \cdot \hat{z}|$$

If we define $R = \dfrac{|\langle \vec{S}_R \rangle \cdot \hat{z}|}{|\langle \vec{S}_I \rangle \cdot \hat{z}|}$, $T = \dfrac{|\langle \vec{S}_T \rangle \cdot \hat{z}|}{|\langle \vec{S}_I \rangle \cdot \hat{z}|}$

Then we get

$$1 = T + R$$

Also

$$R = \frac{\frac{|k_R|}{2\omega\mu_a} |\vec{E}_{R\omega}|^2 \cos\theta_I}{\frac{|k_I|}{2\omega\mu_a} |\vec{E}_{I\omega}|^2 \cos\theta_I} = \frac{|\vec{E}_{R\omega}|^2}{|\vec{E}_{I\omega}|^2}$$ since $|k_I| = |k_R|$

But

$$T = \frac{\frac{|k_T|}{2\omega\mu_b} |\vec{E}_{T\omega}|^2 \cos\theta_T}{\frac{|k_I|}{2\omega\mu_a} |\vec{E}_{I\omega}|^2 \cos\theta_I} \neq \frac{|\vec{E}_{T\omega}|^2}{|\vec{E}_{I\omega}|^2}$$

Radiation from moving charges

In Lorentz gauge: $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$

potentials solve $\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \equiv \Box^2 V = -\rho/\varepsilon_0$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t} \equiv \Box^2\vec{A} = -\mu_0\vec{J}$$

if we know potentials, can get fields from

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad , \quad \vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

As in electro and magneto statics, it is easier to solve for V and $\vec{A}$ and then determine $\vec{E}$ and $\vec{B}$, rather than try to solve for $\vec{E}$ and $\vec{B}$ directly.

Recall solutions for statics: $\nabla^2 V = -\rho/\varepsilon_0$, $\nabla^2\vec{A} = -\mu_0\vec{J}$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(r')}{|r-r'|} \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')}{|r-r'|}\,d^3r'$$

Both solutions follow from the fact that

$$\nabla^2\left(\frac{1}{|r-r'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}\,')$$

↑ Dirac δ-function

$\frac{1}{|r-r'|}$ is called the "Green's function" for the operator ∇^2

Similarly, if we could find the "Green's function" for the $\Box^2$ operator, ie a function $G(\vec{r}-\vec{r}', t-t')$ that solved

$$\Box^2 G(\vec{r}-\vec{r}', t-t') = -4\pi\,\delta(\vec{r}-\vec{r}')\,\delta(t-t')$$

then the solutions to $\Box^2 V = -\rho/\varepsilon_0$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$, would be

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\, G(\vec{r}-\vec{r}', t-t')\,\rho(\vec{r}',t')$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\, G(\vec{r}-\vec{r}', t-t)\,\vec{J}(r',t')$$

How to find $G(\vec{r}-\vec{r}', t-t')$? Use Fourier transf method

$$G(\vec{r},t) = \int d^3k \int d\omega\, \tilde{G}(\vec{k},\omega)\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}$$

$$\delta(\vec{r}) = \int d^3k\, \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3} \qquad \text{from HW}$$

$$\delta(t) = \int d\omega\, \frac{e^{-i\omega t}}{2\pi}$$

substitute into $\Box^2 G(\vec{r},t) = -4\pi\,\delta(\vec{r})\,\delta(t)$

$$\int d^3k \int d\omega\, \tilde{G}(k,\omega)\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t} = -4\pi \int d^3k \int d\omega\, \frac{e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}}{(2\pi)^4}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\int d^3k \int d\omega \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \left(\frac{\omega^2}{c^2} - k^2 \right) \tilde{G}(k,\omega) = \int d^3k \int d\omega \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \frac{(-4\pi)}{(2\pi)^4}$$

equate Fourier coefficients

$$\Rightarrow \left(\frac{\omega^2}{c^2} - k^2 \right) \tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4}$$

$$\tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4} \frac{c^2}{(\omega^2 - c^2k^2)}$$

$$G(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tilde{G}(k,\omega)$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{(2\pi)^4} \frac{(-) 4\pi c^2}{(\omega+ck)(\omega-ck)}$$

integrand diverges when $\omega = \pm ck$

Can evaluate using methods of complex contour integration (see complex variables course)

$$G(\vec{r},t) = \begin{cases} 0 & t < 0 \\ \frac{c}{r}\delta(r-ct) & t > 0 \end{cases} \quad \text{where } r = |\vec{r}|$$

$$\boxed{G(\vec{r},t) = \begin{cases} 0 & t < 0 \\ \frac{1}{r}\delta\left(t - \frac{r}{c}\right) & t > 0 \end{cases}} \quad \text{as } \delta(ax) = \frac{\delta(x)}{a}$$

$G(\vec{r},t)$ has a reasonable form:

1) $G(\vec{r},t) \sim \delta(|\vec{r}|-ct)$ $\Rightarrow$ response travels with speed c

response from source at $\vec{r}\,'=0$, $t'=0$, is only felt at ~~time t = r~~ position $\vec{r}$ at time $t = \frac{r}{c}$ later.

2) If take $c \to \infty$, $G(\vec{r},t) \to \frac{\delta(t)}{r}$ response instantaneous and $\frac{1}{r}$ is Greens function of ∇^2

expected as $\lim_{c\to\infty} \Box^2 = \nabla^2$.

Explicit check that $G = \frac{c}{r}\delta(r-ct)$ solves $\Box^2 G = -4\pi\,\delta(\vec{r})\,\delta(t)$

$$\nabla^2(ab) = a\nabla^2 b + b\nabla^2 a + 2\vec{\nabla}a\cdot\vec{\nabla}b$$

$$\nabla^2\left[\frac{c}{r}\delta(r-ct)\right] = \frac{c}{r}\nabla^2\delta(r-ct) + 2\,\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) + \delta(r-ct)\,\nabla^2\left(\frac{c}{r}\right)$$

use $\nabla^2\delta(r-ct) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\delta(r-ct)\right)$ in spherical coords

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\delta'(r-ct)\right) = \frac{1}{r^2}\left[2r\,\delta'(r-ct) + r^2\delta''(r-ct)\right]$$

$$= \frac{2}{r}\delta'(r-ct) + \delta''(r-ct)$$ here $\delta'(x) = \frac{d\delta(x)}{dx}$ etc.

$$\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) = \left(\frac{-c}{r^2}\right)\left(\delta'(r-ct)\right)$$ (evaluated in spherical coords)

$$\nabla^2\left(\frac{c}{r}\right) = -4\pi\,\delta(\vec{r})\,c$$

so $\nabla^2\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{c}{r}\left(\frac{2}{r}\delta'(r-ct) + \delta''(r-ct)\right)$

$$- \frac{2c}{r^2}\,\delta'(r-ct) \quad - 4\pi c\,\delta(\vec{r})\,\delta(r-ct)$$

$$= \frac{c}{r}\,\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta\!\left(t - \frac{r}{c}\right)$$ using $\delta(r-ct) = \frac{\delta(t-\frac{r}{c})}{c}$

$$= \frac{c}{r}\,\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta(t)$$

↑ since $r=0$ because of $\delta(r)$ term

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{c}{r}(-c)\,\delta'(r-ct)\right]$$

$$= -\frac{1}{r}\frac{\partial}{\partial t}\,\delta'(r-ct) \;=\; \frac{c}{r}\,\delta''(r-ct)$$

so $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{c}{r}\,\delta(r-ct)\right) = \frac{c}{r}\delta''(r-ct) - 4\pi\,\delta(\vec{r})\,\delta(t)$

$$- \frac{c}{r}\,\delta''(r-ct)$$

$$= -4\pi\,\delta(\vec{r})\,\delta(t)$$ as desired

$$V(\vec{r},t) = \int \frac{d^3r'}{4\pi\varepsilon_0}\;\frac{\rho(r',t')}{|\vec{r}-\vec{r}'|}$$

where $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$ "retarded time"
depends on $\vec{r}$ and $\vec{r}'$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\;\frac{\vec{J}\left(\vec{r}',\, t' = t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

general solution to inhomogeneous wave equation