

Summary

The Green's function solves

$$\square^2 G(\vec{r}-\vec{r}', t-t') = -4\pi \delta(\vec{r}-\vec{r}') \delta(t-t')$$

Therefore the solution to

$$\square^2 V(\vec{r}, t) = -\frac{\rho(\vec{r}, t)}{\epsilon_0}$$

is

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \rho(\vec{r}', t') G(\vec{r}-\vec{r}', t-t')$$

check:

$$\square^2 V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \rho(\vec{r}', t') \square^2 G(\vec{r}-\vec{r}', t-t')$$

↑ since $\square^2$ acts on $\vec{r}$ and t
not on $\vec{r}'$ and t'

$$= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \rho(\vec{r}', t') (-4\pi) \delta(\vec{r}-\vec{r}') \delta(t-t')$$

integrate over the δ -functions

$$= -\frac{\rho(\vec{r}, t)}{\epsilon_0}$$

Similarly, the solution to $\square^2 \vec{A}(\vec{r}, t) = \mu_0 \vec{j}(\vec{r}, t)$ is

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \vec{j}(\vec{r}', t') G(\vec{r}-\vec{r}', t-t')$$

Now our solution for the Green's function was

$$G(\vec{r}-\vec{r}', t-t') = \begin{cases} 0 & t < t' \\ \frac{1}{|\vec{r}-\vec{r}'|} \delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c}) & t > t' \end{cases}$$

So we can write our solution for $V(\vec{r}, t)$ as

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^t dt' \rho(\vec{r}', t') \frac{1}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)$$

do integral over the δ -function

$$= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} d^3r' \frac{\rho(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c})}{|\vec{r} - \vec{r}'|}$$

$$\text{or } V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} d^3r' \frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|}$$

where here t' is the "retarded" time

$$t' = t - \frac{|\vec{r} - \vec{r}'|}{c} \quad \text{depends on } \vec{r}'$$

Similarly

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} d^3r' \frac{\vec{j}(\vec{r}', t')}{|\vec{r} - \vec{r}'|}$$

$$t' = t - \frac{|\vec{r} - \vec{r}'|}{c}$$

Radiation by localized oscillating charge distribution

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' dt' \vec{j}(\vec{r}', t') \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}$$

For pure harmonic oscillating current $\vec{j}(\vec{r}, t) = \text{Re} \left\{ \vec{j}(\vec{r}, \omega) e^{-i\omega t} \right\}$
 Resulting $\vec{A}$ will oscillate at same freq ω $\vec{A}(\vec{r}, t) = \text{Re} \left\{ \vec{A}(\vec{r}, \omega) e^{-i\omega t} \right\}$

$$\vec{A}(\vec{r}, \omega) e^{-i\omega t} = \frac{\mu_0}{4\pi} \int d^3r' dt' \vec{j}(\vec{r}', \omega) e^{-i\omega t'} \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}$$

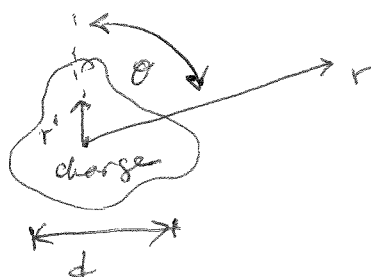
$$\vec{A}(\vec{r}, \omega) e^{-i\omega t} = \frac{\mu_0}{4\pi} \int d^3r' \vec{j}(\vec{r}', \omega) e^{-i\omega t} \frac{e^{+i\omega \frac{|\vec{r}-\vec{r}'|}{c}}}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \vec{j}(\vec{r}', \omega) \frac{e^{i\omega |\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume $\vec{j}(\vec{r}', \omega) \approx 0$ for $|\vec{r}'| > d$, i.e. charge is localized within region of size d about origin.

Approx ①

for $r \gg d$, i.e. far from sources



$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2rr' \cos \theta}$$

$$= r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r} \cos \theta}$$

$$\approx r \left(1 - \frac{r'}{r} \cos \theta + o\left(\left(\frac{r'}{r}\right)^2\right) \right)$$

$$\approx r - \vec{r}' \cdot \hat{r}$$

$\hat{r}$ is unit vector along $\vec{r}$

$$\vec{A}(\vec{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}', \omega) e^{ik(r - \vec{r}' \cdot \hat{r})}}{r - \vec{r}' \cdot \hat{r}} \quad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' \frac{\vec{j}(\vec{r}', \omega) e^{-ik\vec{r}' \cdot \hat{r}}}{1 - \frac{\hat{r} \cdot \vec{r}'}{r}} \quad \leftarrow \text{expand } \frac{1}{1-\delta} \sim 1+\delta$$

$$= \frac{\mu_0}{4\pi} \left(\frac{e^{ikr}}{r} \right) \int d^3r' \vec{j}(\vec{r}', \omega) e^{-ik\hat{r} \cdot \vec{r}'} \left(1 + \frac{\hat{r} \cdot \vec{r}'}{r} \right)$$

$\uparrow$
 when combine with factor $e^{-i\omega t}$ ($\vec{A}(\vec{r}, t) = \vec{A}(\vec{r}, \omega) e^{-i\omega t}$)
 this piece gives spherical waves

$$\frac{e^{i(kr - \omega t)}}{r}$$

$\Rightarrow$ oscillating charge radiates outgoing spherical electromagnetic waves

$\int d^3r \vec{j}$ --- term will determine angular dependence of the radiation

$$\vec{A}(\vec{r}, \omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' \vec{j}(\vec{r}', \omega) e^{-ik\hat{r} \cdot \vec{r}'} \left(1 + \frac{\hat{r} \cdot \vec{r}'}{r}\right)$$

Approximation ② long wave length limit $\lambda \gg d$
 or $kd \ll 1 \Rightarrow \frac{\omega}{c}d \ll 1$ or $\frac{d}{c} \ll 1$

the maximum
 Since d is distance over which charge moves in period of oscillation τ , we see that $kd \ll 1$

$\Rightarrow v \ll c$ where $v \approx \frac{d}{\tau}$ is characteristic velocity with which the charges move
 $\Rightarrow \lambda \gg d$ is a non-relativistic approx.

$$kd \ll 1 \Rightarrow e^{-ik\hat{r} \cdot \vec{r}'} \approx 1 - ik\hat{r} \cdot \vec{r}' + \text{higher order}$$

$$\vec{A}(\vec{r}, \omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' \vec{j}(\vec{r}', \omega) \underbrace{\left(1 - ik\hat{r} \cdot \vec{r}'\right)\left(1 + \frac{\hat{r} \cdot \vec{r}'}{r}\right)}_{\left[1 + \hat{r} \cdot \vec{r}'\left(\frac{1}{r} - ik\right)\right]}$$

+ higher order terms
 in $\frac{d}{r}$ or kd .

$$\vec{A}(\vec{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left\{ \vec{I}_1 + \left(\frac{1}{r} - ik\right) \vec{I}_2 \right\}$$

where $\vec{I}_1 \equiv \int d^3r' \vec{j}(\vec{r}', \omega)$

$$\vec{I}_2 \equiv \int d^3r' \hat{r} \cdot \vec{r}' \vec{j}(\vec{r}', \omega)$$

Now evaluate $\vec{I}_1$ and $\vec{I}_2$

i th component of $\vec{I}_1$

$$I_{1i} = \int d^3r' \dot{j}_i(\vec{r}', \omega) = - \int d^3r' r'_i \vec{\nabla}' \cdot \vec{j} \quad \text{via integration by parts}$$

boundary term vanishes as $\vec{j} \rightarrow 0$ far from source

to see this use: ^{trick} ① $\vec{\nabla} \cdot (f \vec{g}) - \vec{g} \cdot \vec{\nabla} f = f \vec{\nabla} \cdot \vec{g}$ product rule
 apply to right hand side with $f \equiv r'_i$, $\vec{g} \equiv \vec{j}$

$$\Rightarrow - \int d^3r' r'_i \vec{\nabla}' \cdot \vec{j} = - \int d^3r' \left\{ \vec{\nabla}' \cdot (r'_i \vec{j}) - \vec{j} \cdot \vec{\nabla}' r'_i \right\}$$

first term: ^{trick} ② $\int d^3r' \vec{\nabla}' \cdot (r'_i \vec{j}) = \oint d\vec{a} \cdot (r'_i \vec{j}) = 0$ as $\vec{j}(r' \rightarrow \infty) = 0$
surface $\rightarrow \infty$ as $\vec{j}$ is localized

second term: ^{trick} ③ $\vec{\nabla}' r'_i =$ unit vector in direction i
 to see this, consider example: $\vec{\nabla} x = \frac{\partial x}{\partial x} \hat{x} + \frac{\partial x}{\partial y} \hat{y} + \frac{\partial x}{\partial z} \hat{z} = \hat{x} + 0 + 0$
 $\Rightarrow \vec{j} \cdot \vec{\nabla}' r'_i = \dot{j}_i$

put together: $- \int d^3r' r'_i \vec{\nabla}' \cdot \vec{j} = 0 + \int d^3r' \dot{j}_i = I_{1i}$ as desired

Now use ^{trick} ④ charge conservation $\Rightarrow \vec{\nabla}' \cdot \vec{j} = - \frac{\partial \rho}{\partial t} = i\omega \rho(\vec{r}, \omega)$
 since $\rho(\vec{r}, t) = \rho(\vec{r}, \omega) e^{-i\omega t}$

$$\vec{I}_1 = - \int d^3r' \vec{r}' \vec{\nabla}' \cdot \vec{j} = -i\omega \int d^3r' \vec{r}' \rho(\vec{r}', \omega)$$

$\vec{p}(\omega)$ electric dipole moment

$$\vec{I}_1 = -i\omega \vec{p}(\omega)$$

i^{th} component of $\vec{I}_2$

$$I_{2i} = \int d^3r' (\hat{r} \cdot \vec{r}') \vec{f}_i = \int d^3r' (\hat{r} \cdot \vec{r}') (\vec{f} \cdot \vec{\nabla}' r'_i) \text{ by trick ③}$$

$$= \sum_{k=1}^3 \hat{r}_k \int d^3r' (r'_k \vec{f}) \cdot \vec{\nabla}' r'_i \quad \text{writing out } \hat{r} \cdot \vec{r}' = \sum_{k=1}^3 \hat{r}_k r'_k$$

as sum over components

$$= \sum_{k=1}^3 \hat{r}_k \int d^3r' \left\{ \underbrace{\vec{\nabla}' \cdot (r'_i r'_k \vec{f})}_{=0 \text{ by trick ②}} - r'_i \vec{\nabla}' \cdot (r'_k \vec{f}) \right\} \quad \text{by trick ①}$$

with $\begin{cases} \vec{f} \equiv r'_i \\ \vec{g} \equiv r'_k \vec{f} \end{cases}$

$$= - \sum_{k=1}^3 \hat{r}_k \int d^3r' \left[\underbrace{r'_i r'_k \vec{\nabla}' \cdot \vec{f}}_{i\omega r'_i r'_k \rho \text{ by trick ④}} + \underbrace{r'_i \vec{f} \cdot \vec{\nabla}' r'_k}_{r'_i \vec{f}_k \text{ by trick ③}} \right] \quad \text{expanding } \vec{\nabla}' \cdot (r'_k \vec{f}) \text{ as in trick ①}$$

$$= - \sum_{k=1}^3 \hat{r}_k \int d^3r' [r'_i \vec{f}_k + i\omega r'_i r'_k \rho]$$

Last trick: $I_{2i} = \frac{1}{2} I_{2i} + \frac{1}{2} I_{2i}$

$$= \frac{1}{2} \sum_k \hat{r}_k \left[\underbrace{\int d^3r' r'_k \vec{f}_i}_{\text{from definition of } I_{2i}} - \underbrace{\int d^3r' \{ r'_i \vec{f}_k + i\omega r'_i r'_k \rho \}}_{\text{from above manipulations}} \right]$$

$$\vec{I}_2 = \frac{1}{2} \int d^3r' \left[\underbrace{(\hat{r} \cdot \vec{r}') \vec{f} - (\hat{r} \cdot \vec{f}) \vec{r}'}_{-\hat{r} \times (\vec{r}' \times \vec{f}) \text{ triple product rule}} \right] - \frac{1}{2} \int d^3r' i\omega \hat{r} \cdot \vec{r}' \vec{r}' \rho$$

$$= -\frac{1}{2} \hat{r} \times \int d^3r' (\vec{r}' \times \vec{f}) - \frac{1}{2} i\omega \hat{r} \cdot \int d^3r' (\vec{r}' \vec{r}') \rho$$

$$= -\hat{r} \times \vec{m}(\omega) - \frac{1}{2} \frac{i\omega}{3} \hat{r} \cdot \vec{Q}'(\omega)$$

where $\vec{m}(\omega) \equiv \frac{1}{2} \int d^3r' (\vec{r}' \times \vec{j}(\vec{r}', \omega))$ is magnetic dipole moment

$$\vec{Q}'_{ij}(\omega) = \int d^3r' 3 \vec{r}'_i \vec{r}'_j \rho(\vec{r}', \omega)$$

looks very close to electric quadrupole tensor

$$\vec{Q}_{ij} = \int d^3r' (3 \vec{r}'_i \vec{r}'_j - r'^2 \delta_{ij}) \rho(\vec{r}', \omega)$$

$$\vec{Q}'_{ij} = \vec{Q}_{ij} + \delta_{ij} \int d^3r' r'^2 \rho(\vec{r}', \omega)$$

$$\vec{I}_2 = -\hat{r} \times \vec{m}(\omega) - \frac{i\omega}{6} \hat{r} \cdot \vec{Q}(\omega) + \frac{i\omega}{6} \hat{r} \underbrace{\int d^3r' r'^2 \rho(\vec{r}', \omega)}_{\text{call this } \mathcal{C}(\omega) \text{ a scalar}}$$

plug back into $\vec{A}(\vec{r}, \omega)$

$$\vec{A}(\vec{r}, \omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left\{ \vec{I}_1 + \left(\frac{1}{r} - ik \right) \vec{I}_2 \right\}$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left\{ \underbrace{-i\omega \vec{p}}_{\text{electric dipole contribution}} - \left(\frac{1}{r} - ik \right) \left(\underbrace{\hat{r} \times \vec{m}}_{\text{magnetic dipole contribution}} + \frac{i\omega}{6} \hat{r} \cdot \underbrace{\vec{Q}}_{\text{electric quadrupole contribution}} + \frac{i\omega}{6} \hat{r} \mathcal{C} \right) \right\}$$