The last piece which contributes to $\vec{A}$, ie $\frac{i\omega}{6} c \hat{r} \frac{e^{ikr}}{r}$ is unimportant - it does not effect the $\vec{E}$ or $\vec{B}$ fields since

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{\nabla} \times [f(r)\hat{r}] = 0$$

similarly, away from sources, where $\vec{J} = 0$, Ampere's law gives

$$\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

$$\parallel$$

$$-i\omega\mu_0\varepsilon_0 \vec{E}(\vec{r},\omega) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

since last term doesn't contribute to $\vec{B}$, it doesn't contribute to $\vec{E}$. Formally, we could remove it by making a gauge transformation. Less formally, we will just drop it!

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left\{ -i\omega\vec{p} - \overbrace{\left(\frac{1}{r} - ik\right)}^{= -\left(1 + \frac{i}{kr}\right)ik} \left(\hat{r} \times \vec{m} + \frac{i\omega}{6} \hat{r} \cdot \overleftrightarrow{Q} \right) \right\}$$

lets look at relative strengths of the different terms

far from sources, $\frac{1}{r}$ will be small compared to k.

radiation zone: just consider those terms in $\vec{A}$ that decrease as slowest powers of $\left(\frac{1}{r}\right)^n$. This will be the $\frac{1}{r}$ terms

Approx ① $d \ll r$

Approx ② $d \ll \lambda$

Radiation Zone $\lambda \ll r$ so $kr \gg 1$

Combine: $d \ll \lambda \ll r$ is RZ

lectric dipole term $\quad \vec{p} \simeq q\,d \quad$ q is typical charge in source
d is size of source region

magnetic dipole term $\quad \vec{m} = \frac{1}{2}\int d^3r\ \vec{r}\times\vec{J} \quad$ $\vec{J} \sim q v$ where v is typical velocity

$$\simeq d\,j \simeq d\,v\,q \qquad v \sim \frac{d}{\tau} \sim d\omega \sim dck$$

$$\simeq q\,d^2\omega \sim q c d^2 k$$

electric quadrapole term $\overleftrightarrow{Q} \sim \int d^3r\ \vec{r}\,\vec{r}\,\rho$

$$\sim q d^2$$

so electric dipole contrib to $\vec{A}$ goes as $\quad \omega\vec{p} \sim q\omega d = qc(kd)$
magnetic dipole contrib to $\vec{A}$ goes as $\quad k\vec{m} \sim q\omega k d^2 = qc(kd)^2$
electric quadrapole contrib to $\vec{A}$ goes as $\quad k\omega\overleftrightarrow{Q} \sim q\omega k d^2 = qc(kd)^2$

Since approx ② assumed (kd) was small
(non relativistic approx: $kd \approx v/c$)
we have an expansion for $\vec{A}$ in powers of (kd)

leading term is the electric dipole term.
next order terms are $\left\{\begin{array}{l}\text{magnetic dipole}\\ \text{electric quadrapole}\end{array}\right\}$ ← these are comparable in strength.

If we kept higher order terms in our expansion, the next terms would be the magnetic quadrapole and electric octopole, both of order $qc(kd)^3$.

Consider now the leading term, the electric dipole term

$$\vec{A}_{E1} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-i\omega)\,\vec{p}(\omega) \qquad \text{"}E1\text{"} \equiv \text{electric dipole term}$$

<u>magnetic field</u>

$$\vec{B}_{E1}(\vec{r},\omega) = \vec{\nabla}\times\vec{A}_{E1}(\vec{r},\omega) = \frac{-i\omega\mu_0}{4\pi}\,\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\vec{p}(\omega)\right)$$

use $\vec{\nabla}\times(f\vec{g}) = (\vec{\nabla}f)\times\vec{g} + f\,\vec{\nabla}\times\vec{g}$ with $f = e^{ikr}$, $\vec{g} = \frac{\vec{p}(\omega)}{r}$

$$\vec{\nabla}e^{ikr} = \hat{r}\frac{\partial}{\partial r}(e^{ikr}) = e^{ikr}\,ik\hat{r} \qquad \text{in spherical coordinates}$$

$$\vec{\nabla}\times\left(\frac{\vec{p}}{r}\right) = \left(\vec{\nabla}\frac{1}{r}\right)\times\vec{p} + \underbrace{\frac{1}{r}\vec{\nabla}\times\vec{p}}_{=0 \text{ since } \vec{p} \text{ is constant}} = -\frac{1}{r^2}\hat{r}\times\vec{p}$$

$$\vec{B}_{E1}(\vec{r},\omega) = \frac{-i\omega\mu_0}{4\pi}\left[e^{ikr}\,ik\,\frac{\hat{r}\times\vec{p}(\omega)}{r} - e^{ikr}\frac{\hat{r}\times\vec{p}(\omega)}{r^2}\right]$$

use $\omega = ck$ $\quad \boxed{\vec{B}_{E1} = -\frac{c\mu_0 k^2}{4\pi}\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\vec{p}(\omega)\times\hat{r}}$ $\quad$ use $\hat{r}\times\vec{p} = -\vec{p}\times\hat{r}$

small compared to 1 when $kr \gg 1$

We define the <u>Radiation Zone</u> limit when $r \gg \lambda \Rightarrow kr \gg 1$
far away on the scale of the wavelength of the radiated wave.
In this limit the 2nd term is small compared to the first
$(1 + i/kr) \approx 1$ and we have

$$\text{in R.Z.} \quad \boxed{\vec{B}_{E1}(\vec{r},\omega) = -\frac{c\mu_0 k^2}{4\pi}\frac{e^{ikr}}{r}\vec{p}(\omega)\times\hat{r}}$$

electric field

from Ampere, $\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B}$ since $\vec{J} = 0$ far from source

$$\Rightarrow \vec{E}_{E1} = \frac{i}{\omega \mu_0 \varepsilon_0} \vec{\nabla} \times \vec{B}_{E1} \quad \text{since } \frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E}$$

$$= \frac{-i}{\omega \mu_0 \varepsilon_0} \frac{c\mu_0}{4\pi} k^2 \, \vec{\nabla} \times \left(\frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

$\omega = ck$

to evaluate the $\vec{\nabla} \times (\ )$ term, use $\vec{\nabla} \times (f\vec{g}) = f \vec{\nabla} \times \vec{g} + \vec{\nabla} f \times \vec{g}$

with $f = e^{ikr}$, $\vec{g} = \frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r}$

then $$\vec{\nabla} \times (\ ) = \left(\vec{\nabla} e^{ikr}\right) \times \left(\frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

$$+ e^{ikr} \vec{\nabla} \times \left(\frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

But in the radiation zone we can ignore the second term, since

$$\vec{\nabla} \times \left(\frac{1}{r}\left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right) \sim \frac{1}{r^2}$$

we see this by noting that

$$\vec{\nabla} \frac{1}{r} \sim \frac{1}{r^2}, \quad \vec{\nabla} \frac{1}{r^2} \sim \frac{1}{r^3}$$

$$\vec{\nabla} \hat{r} \sim \frac{\partial}{\partial x} \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{x}}{|\vec{r}|} - \frac{\vec{r}}{|\vec{r}|^2} \frac{x}{|\vec{r}|} = \frac{\hat{x}}{|\vec{r}|} - \frac{x\hat{r}}{|\vec{r}|^2} \sim O\left(\frac{1}{r}\right)$$

So keep only 1st term in Radiation Zone and we get

$$\vec{E}_{E1} = \frac{-i}{\omega \mu_0 \varepsilon_0} \frac{c\mu_0}{4\pi} k^2 \underbrace{\left(ike^{ikr} \hat{r} \right)}_{= \vec{\nabla} e^{ikr}} \times \left(\frac{1}{r}\left(1 + \underbrace{\frac{i}{kr}}_{\text{ignore in RZ}}\right) \vec{p} \times \hat{r} \right)$$

use $\omega = ck$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} \, \hat{r} \times (\vec{p} \times \hat{r})}$$

electric field - full calculation without making Radiation Zone approximation

from Ampere, $\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B}$, since $\vec{J} = 0$ for far from source

$$\Rightarrow \quad \vec{E}_{E1} = \frac{i}{\omega \mu_0 \varepsilon_0} \vec{\nabla} \times \vec{B}_{E1} \qquad \text{since } \frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E}$$

$$= \frac{-i}{\omega \mu_0 \varepsilon_0} \frac{c\mu_0}{4\pi} k^2 \vec{\nabla} \times \left(\frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \vec{p} \times \hat{r} \right)$$

$\omega = ck$

to evaluate $\vec{\nabla} \times (\;)$, use $\vec{\nabla} \times (f\vec{g}) = f\, \vec{\nabla} \times \vec{g} + \vec{\nabla} f \times \vec{g}$

with $f = \frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)$ and $\vec{g} = \vec{p} \times \hat{r}$

$$\vec{\nabla} \times (\;) = \frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right) \vec{\nabla} \times (\vec{p} \times \hat{r}) + \vec{\nabla}\left(\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\right) \times (\vec{p} \times \hat{r})$$

evaluate second term: $\vec{\nabla}\left(\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\right) = \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\right) \hat{r}$ in spherical coords

$$= e^{ikr}\left[ik\left(\frac{1}{r} + \frac{i}{kr^2}\right) - \frac{1}{r^2} - \frac{2i}{kr^3} \right] \hat{r}$$

$$= \frac{e^{ikr}}{r}\left[ik - \frac{2}{r} - \frac{2i}{kr^2} \right] \hat{r}$$

evaluate first term: $\vec{\nabla} \times (\vec{p} \times \hat{r}) = \vec{p}\,(\vec{\nabla} \cdot \hat{r}) - (\vec{p} \cdot \vec{\nabla})\hat{r}$ cover Griffiths

where $\vec{\nabla} \cdot \hat{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2}{r}$ evaluating in spherical coordinates

and $(\vec{p} \cdot \vec{\nabla})\hat{r} = \sum_k p_k \frac{\partial \hat{r}}{\partial r_k}$

unit vector in k direction

where $\frac{\partial \hat{r}}{\partial r_k} = \frac{\partial}{\partial r_k}\left(\frac{\vec{r}}{r}\right) = \vec{r}\left(-\frac{1}{r^2}\frac{\partial r}{\partial r_k}\right) + \frac{\hat{e}_k}{r}$

$$= \vec{r}\left(-\frac{1}{r^2}\frac{r_k}{r}\right) + \frac{\hat{e}_k}{r} \qquad \text{as } \frac{\partial r}{\partial r_k} = \frac{r_k}{r}$$

so $\quad \vec{\nabla}\times(\vec{p}\times\hat{r}) = \frac{2\vec{p}}{r} - \sum_k p_k \left(-\frac{\vec{r}\, r_k}{r^3} + \frac{\hat{e}_k}{r} \right)$

$$= \frac{2\vec{p}}{r} + \frac{\vec{r}\ \vec{p}\cdot\vec{r}}{r^3} - \frac{\vec{p}}{r}$$

$$= \frac{\vec{p} + \hat{r}(\vec{p}\cdot\hat{r})}{r} \qquad \text{using } \hat{r} = \frac{\vec{r}}{r}$$

putting all the pieces together

$$\vec{E}_{E1} = \frac{-ik}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\left(1+\frac{i}{kr}\right)\frac{\vec{p}+\hat{r}(\vec{p}\cdot\hat{r})}{r} + \left(ik - \frac{2}{r} - \frac{2i}{kr^2}\right)\underbrace{\hat{r}\times(\vec{p}\times\hat{r})}_{\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})}\right]$$

order by powers of $\frac{1}{r}$

$$= -\frac{ik}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[ik(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})) + \frac{1}{r}\left(1+\frac{i}{kr}\right)(\vec{p}+\hat{r}(\vec{p}\cdot\hat{r})) - \frac{2}{r}\left(1+\frac{i}{kr}\right)(\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}))\right]$$

$$= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(\vec{p}+\hat{r}(\vec{p}\cdot\hat{r}) - 2\vec{p} + 2\hat{r}(\vec{p}\cdot\hat{r})\right)\right]$$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(3\hat{r}(\vec{p}\cdot\hat{r})-\vec{p}\right)\right]}$$

<u>radiation zone approx</u> $\quad kr \gg 1 \quad$ keep only terms of order $\frac{1}{r}$

$$\boxed{\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\underbrace{\left[\vec{p}-\hat{r}(\vec{p}\cdot\hat{r})\right]}_{\hat{r}\times(\vec{p}\times\hat{r})}}$$

Radiation zone limit

$$\vec{E}_{E1} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\,\hat{r}\times(\vec{p}\times\hat{r})$$

$$\vec{B}_{E1} = -\frac{c\mu_0}{4\pi}k^2\frac{e^{ikr}}{r}\,\vec{p}\times\hat{r}$$

radiation zone fields in electric dipole approx

$\vec{E}$ and $\vec{B}$ are outwards traveling spherical waves $\sim \dfrac{e^{ikr}}{r}$

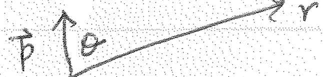
using $\mu_0\varepsilon_0 = 1/c^2$

$$\frac{|\vec{B}_{E1}|}{|\vec{E}_{E1}|} = \frac{c\mu_0}{4\pi}\cdot 4\pi\varepsilon_0 = c\mu_0\varepsilon_0 = \frac{c}{c^2} = \frac{1}{c}$$

just as for plane waves in vacuum

---

if choose coordinates so that $\vec{p}$ is along $\hat{z}$ axis



$\vec{p}\times\hat{r} = -\hat{\phi}\sin\theta\, p$

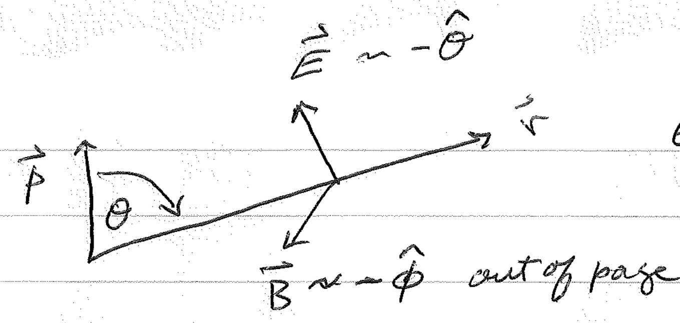
$\hat{r}\times(\vec{p}\times\hat{r}) = p\sin\theta(\hat{r}\times\hat{\phi})$

$= -\hat{\theta}\sin\theta\, p$

$$\vec{E}_{E1} = \frac{-k^2 p}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\sin\theta\,\hat{\theta}$$

$$\vec{B}_{E1} = -\frac{c\mu_0}{4\pi}k^2 p\,\frac{e^{ikr}}{r}\sin\theta\,\hat{\phi}$$

$\hat{\theta}$ + $\hat{\phi}$ are spherical coord basis vectors



$\vec{E}$ is in the plane containing $\vec{p}$ and $\hat{r}$
$\vec{B}$ is $\perp$ to this plane

$\vec{E}_{E1}$ and $\vec{B}_{E1}$ are orthogonal, as in a plane wave,
and both are orthogonal to the direction of propagation $\hat{r}$

$\Rightarrow$ oscillating source emits spherical electromagnetic waves

What is Power emitted?

Poynting vector: $\vec{S}_{E1}(\vec{r},t) = \frac{1}{\mu_0} \mathrm{Re}\left[\vec{E}_{E1}(\vec{r},t)\right] \times \mathrm{Re}\left[\vec{B}_{E1}(\vec{r},t)\right]$

$$\mathrm{Re}\left[\vec{E}_{E1}(\vec{r},t)\right] = \mathrm{Re}\left[-\frac{k^2 p}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} \sin\theta\, \hat{\theta}\, e^{-i\omega t}\right]$$

$$= \frac{-k^2 p}{4\pi\varepsilon_0} \frac{\cos(kr-\omega t)}{r} \sin\theta\, \hat{\theta}$$

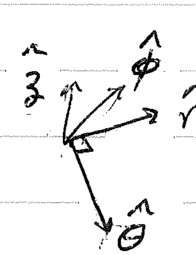
$\leftarrow$ *** Assuming $\vec{p}$ is a real valued vector ***

$$\mathrm{Re}\left[\vec{B}_{E1}(\vec{r},t)\right] = \mathrm{Re}\left[-\frac{c\mu_0}{4\pi} k^2 p \frac{e^{ikr}}{r} \sin\theta\, \hat{\varphi}\, e^{-i\omega t}\right]$$

$$= -\frac{c\mu_0 k^2 p}{4\pi} \frac{\cos(kr-\omega t)}{r} \sin\theta\, \hat{\varphi}$$

$$\vec{S}_{E1} = \frac{1}{\mu_0} \frac{k^2 p}{4\pi\varepsilon_0} \frac{c\mu_0 k^2 p}{4\pi} \frac{\cos^2(kr-\omega t)}{r^2} \sin^2\theta\, (\hat{\theta} \times \hat{\varphi})$$

$$= \frac{c\, k^4 p^2}{(4\pi)^2 \varepsilon_0} \frac{\cos^2(kr-\omega t)}{r^2} \sin^2\theta\, \hat{r}$$



Average over one period of oscillation $\langle \cos^2(kr-\omega t)\rangle = \frac{1}{2}$

$$\langle \vec{S}_{E1} \rangle = \frac{ck^4 p^2}{2(4\pi)^2 \varepsilon_0} \frac{\sin^2\theta}{r^2} \hat{r}$$

Note, the $\frac{1}{r^2}$ is important for energy conservation. If we integrate $\langle \vec{S}_{E1} \rangle \cdot \hat{r}$ over the surface of a sphere of radius r, the result is independent of r.

average energy flux flowing through an element of area at spherical angles θ, ϕ is

$$\text{power } dP_{E1} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle \underbrace{r^2 \sin\theta \, d\theta \, d\phi}$$

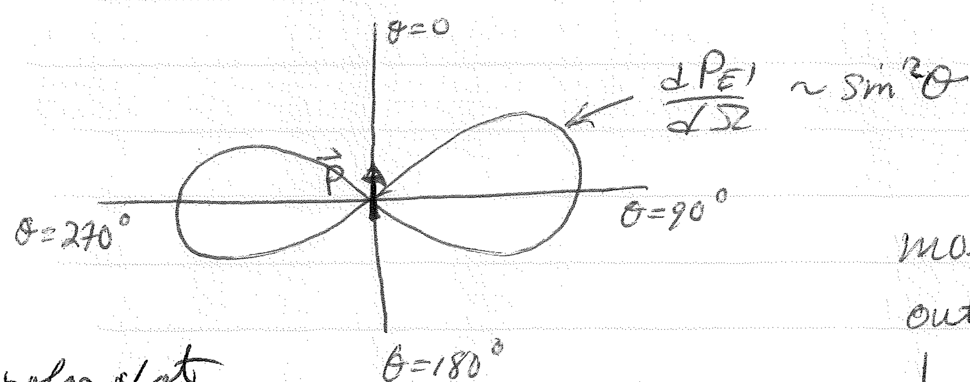
differential area on surface of sphere spanned by $d\theta$ and $d\phi$ $= r^2 d\Omega$

$d\Omega = \sin\theta \, d\theta \, d\phi$ differential solid angle

$$dP_{E1} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle \, r^2 d\Omega$$

$$\frac{dP_{E1}}{d\Omega} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle r^2 = \frac{ck^4 p^2}{2(4\pi)^2 \varepsilon_0} \sin^2\theta$$

$$\frac{dP_{E1}}{d\Omega} = \hat{r}\cdot\langle\vec{S}_{E1}\rangle r^2 = \frac{ck^4p^2}{2(4\pi)^2\varepsilon_0}\sin^2\theta \sim \omega^4\sin^2\theta$$



polar plot

most of power is directed outwards into the plane $\perp$ to $\vec{p}$, ie at angles θ peaked about 90°

For energy conservation to hold, it must be true that all the higher order terms, that go as higher powers of $\frac{1}{r}$ (ie $\frac{1}{r^2}$, $\frac{1}{r^3}$, etc ...), must vanish when compute the time averaged energy flux [integrated over surface of sphere] — otherwise energy would be dissappearing as the wave propagated outwards.

Total power radiated is

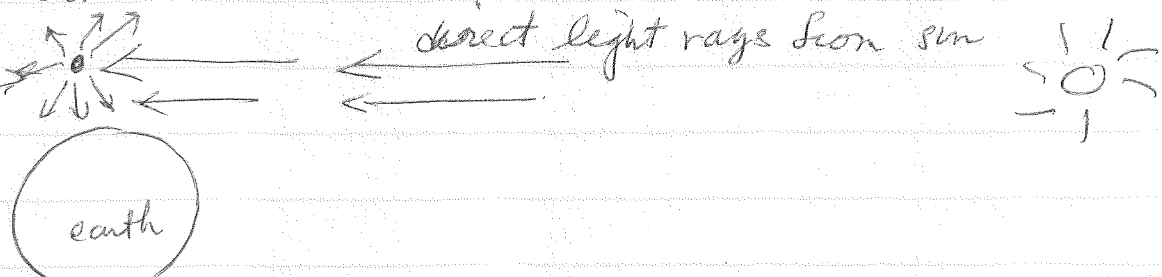
$$P_{E1} = \int \frac{dP_{E1}}{d\Omega}\,d\Omega = \frac{ck^4p^2}{2(4\pi)^2\varepsilon_0}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\,\sin^2\theta$$

$$= \frac{ck^4p^2}{32\pi^2\varepsilon_0}\,2\pi\underbrace{\int_0^{\pi}d\theta\,\sin\theta(1-\cos^2\theta)}_{\left[-\cos\theta+\frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{4}{3}}$$

$$P_{E1} = \frac{ck^4p^2}{4\pi\varepsilon_0\cdot 3} = \boxed{\frac{p^2\omega^4}{4\pi\varepsilon_0\, 3c^3} = P_{E1} \sim \omega^4}$$

Why the sky is blue — Lord Rayleigh

When look at sky, you are seeing the indirect light of the sun, which is the light emitted by the atoms and ~~molect~~ molecules of the atmosphere as they oscilate & so radiate, due to the electric field of the direct light from the sun



molecule, ~~atom~~ in atmosphere oscilates in $\vec{E}$ field of direct rays, and then emitts radiated light with power (can view this as a scattering of the direct rays)

$$P \sim \omega^4 p^2$$

polarizability

p is dipole moment of ~~atom~~ molecule in atmosphere $p = \alpha E$ ← electric field of direct rays

$$\alpha \sim \frac{e^2}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

for molecules in atmosphere, N_2, etc, ω_0 is typically a freq higher than the visible spectrum. Therefore for light in visible spectrum, $\alpha \sim \frac{e^2}{m\omega_0}$ indep of ω

Power emitted $\sim \omega^4$ largest at higher freq.

Since light from sun is "white light" it has components of all freqs. ~~For~~ From the above, we see that this ~~indirect scattered~~ light is most scattered at the higher freqs, due to the ω^4 dependence of

Scattered power in electric dipole approx
$\Rightarrow$ indirect light is strongest in the blue (large ω) part of the visible spectrum. $\Rightarrow$ sky is blue!

When we look at sunrise or sunset however, we are looking at the direct rays of the sun. Since these rays are scattered most in the blue, the direct rays are strongest in the red (small ω) part of the spectrum $\Rightarrow$ sunset & sunrise are red!