<u>Note:</u> when we wrote for $\vec{E}_{E1} = \frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \hat{r}\times(\vec{p}\times\hat{r})$

$$\mathrm{Re}\left[\vec{E}_{E1}(\vec{r},\omega)e^{-i\omega t}\right] = \frac{k^2}{4\pi\epsilon_0} \frac{\cos(kr-\omega t)}{r} \hat{r}\times(\vec{p}\times\hat{r})$$

we implicitly assumed that the amplitude of the oscillating electric dipole moment $\vec{p}(\omega)$ was a real vector. But that is not necessarily always the case!

For $\vec{p}(\omega) \equiv \vec{p}_1$ real, the time dependent dipole moment is

$$\vec{p}(t) = \mathrm{Re}\left[\vec{p}_1 e^{-i\omega t}\right] = \vec{p}_1 \cos\omega t$$

points always in same direction with oscillating magnitude

But suppose $\vec{p}(\omega) = \vec{p}_1 + i\vec{p}_2$. Then

$$\vec{p}(t) = \mathrm{Re}\left[(\vec{p}_1 + i\vec{p}_2)e^{-i\omega t}\right]$$
$$= \vec{p}_1\cos\omega t + \vec{p}_2 \sin\omega t$$

Now <u>direction</u> of $\vec{p}(t)$ is rotating!

if $\vec{p}_1 \perp \vec{p}_2$ then the tip of $\vec{p}(t)$ sweeps out an ellipse! An example of such a $\vec{p}$ would be a charge moving in an elliptical orbit

So if $\vec{p}(\omega)$ is complex we need to be more careful in our calculation of $\vec{S}$

Magnetic Dipole Radiation - in the Radiation Zone Approx

$$\vec{A}_{M1} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} ik\,\hat{r}\times\vec{m}$$

$$\vec{B}_{M1} = \vec{\nabla}\times\vec{A}_{M1} = \frac{\mu_0}{4\pi} ik\,\vec{\nabla}\times\left(e^{ikr}\frac{\hat{r}\times\vec{m}}{r}\right)$$

Exactly the same form as when we computed $\vec{E}_{E1}$ from B_{E1} except $\vec{p}\to\vec{m}$

use $\vec{\nabla}\times(f\vec{g}) = \vec{\nabla}f\times\vec{g} + f\,\vec{\nabla}\times\vec{g}$

$$\vec{B}_{M1} = \frac{\mu_0}{4\pi} ik\left[\underbrace{(\vec{\nabla}e^{ikr})\times\left(\frac{\hat{r}\times\vec{m}}{r}\right)}_{ike^{ikr}\hat{r}\times\left(\frac{\hat{r}\times\vec{m}}{r}\right)} + \underbrace{e^{ikr}\vec{\nabla}\times\left(\frac{\hat{r}\times\vec{m}}{r}\right)}_{\sim O\left(\frac{1}{r^2}\right)}\right]$$

$\sim O\left(\frac{1}{r^2}\right)$ so ignore in RZ approx

$$\vec{B}_{M1} = -\frac{\mu_0}{4\pi} k^2 \frac{e^{ikr}}{r}\,\hat{r}\times(\hat{r}\times\vec{m})$$

$$\vec{E}_{M1} = \frac{i}{\omega\mu_0\varepsilon_0}\vec{\nabla}\times\vec{B}_{M1}$$ from Amperes law with $\vec{J}=0$

$$= \frac{-i\mu_0 k^2}{4\pi\omega\mu_0\varepsilon_0}\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\hat{r}\times(\hat{r}\times\vec{m})\right)$$

$$= \frac{-ik^2}{4\pi\omega\varepsilon_0}\left[\underbrace{(\vec{\nabla}e^{ikr})}_{ik\hat{r}e^{ikr}}\times\left(\frac{\hat{r}\times(\hat{r}\times\vec{m})}{r}\right) + \underbrace{e^{ikr}\vec{\nabla}\times\left(\frac{\hat{r}\times(\hat{r}\times\vec{m})}{r}\right)}_{\sim O\left(\frac{1}{r^2}\right)}\right]$$

$\sim O\left(\frac{1}{r^2}\right)$ so ignore in RZ approx

$$= \frac{k^3}{4\pi\omega\varepsilon_0}\frac{e^{ikr}}{r}\,\hat{r}\times(\hat{r}\times(\hat{r}\times\vec{m}))$$ use $\omega = ck$

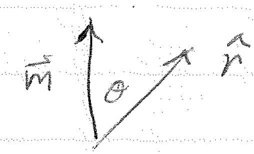
use triple product rule

$$\hat{r}\times(\hat{r}\times(\hat{r}\times\vec{m})) = \hat{r}(\hat{r}\cdot(\hat{r}\times\vec{m})) - (\hat{r}\times\vec{m})(\hat{r}\cdot\hat{r}) = 0 - \hat{r}\times\vec{m}$$

$$\vec{E}_{M1} = \frac{-k^2}{4\pi\varepsilon_0 c}\frac{e^{ikr}}{r}\,\hat{r}\times\vec{m}$$

$$E_{M1} = \frac{k^2}{4\pi\varepsilon_0 c} \frac{e^{ikr}}{r} \left[-\hat{r} \times \vec{m} \right]$$

$$\boxed{\begin{aligned} \vec{E}_{M1} &= \frac{k^2}{4\pi\varepsilon_0 c} \frac{e^{ikr}}{r} \left[\vec{m} \times \hat{r} \right] \\ \vec{B}_{M1} &= \frac{\mu_0 k^2}{4\pi} \frac{e^{ikr}}{r} \left[\hat{r} \times (\vec{m} \times \hat{r}) \right] \end{aligned}}$$



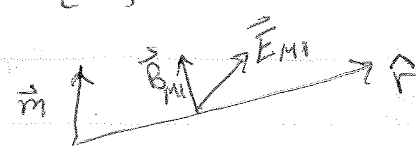
for $\underline{\underline{\vec{m} = m\hat{z}}}$, $\vec{m} \times \hat{r} = m \sin\theta\, \hat{\phi}$

$\hat{r} \times (\vec{m} \times \hat{r}) = m \sin\theta\, (-\hat{\theta})$

$$\left.\begin{aligned} \vec{E}_{M1} &= \frac{k^2 m}{4\pi\varepsilon_0 c} \frac{e^{ikr}}{r} \sin\theta\, \hat{\phi} \\ \vec{B}_{M1} &= -\frac{\mu_0 k^2 m}{4\pi} \frac{e^{ikr}}{r} \sin\theta\, \hat{\theta} \end{aligned}\right\}$$

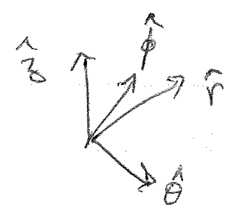
Note similarity with $\vec{E}_{E1}$ and $\vec{B}_{E1}$

$\vec{p} \to \frac{\vec{m}}{c}$, $\vec{E}$ + $\vec{B}$ rotated by $90°$



<u>Poynting vector</u>

$$\cancel{\vec{S}_{M1} = \frac{1}{\mu_0} \vec{E}_{M1}(\vec{r},t) \times \vec{B}_{M1}(\vec{r},t) = \frac{1}{\mu_0} k^2 m}$$



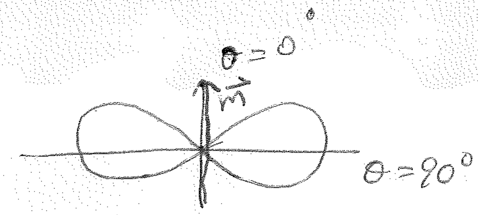
$$\vec{S}_{M1} = \frac{1}{\mu_0} \mathrm{Re}\left\{ \vec{E}_{M1} e^{-i\omega t} \right\} \times \mathrm{Re}\left\{ \vec{B}_{M1} e^{-i\omega t} \right\}$$

$$= \frac{1}{\mu_0} \left(\frac{k^2 m}{4\pi\varepsilon_0 c} \right) \left(-\frac{\mu_0 k^2 m}{4\pi} \right) \frac{\sin^2\theta}{r^2} \cos^2(kr - \omega t) \underbrace{\hat{\phi} \times \hat{\theta}}_{-\hat{r}}$$

$$= \frac{k^4 m^2}{4\pi\varepsilon_0 \;\; 4\pi c} \frac{\sin^2\theta}{r^2} \cos^2(kr - \omega t)\, \hat{r}$$

time average

$$\langle \vec{S}_{M1} \rangle = \frac{k^4 m^2}{32\pi^2 \varepsilon_0 c} \frac{\sin^2\theta}{r^2} \hat{r}$$



power cross section

$$\frac{dP_{M1}}{d\Omega} = \hat{r}\cdot\langle \vec{S}_{M1}\rangle r^2 = \boxed{\frac{k^4 m^2}{2(4\pi)^2 \varepsilon_0 c}\sin^2\theta = \frac{dP_{M1}}{d\Omega}}$$

same form as $\frac{dP_{E1}}{d\Omega}$ with $p \to \frac{m}{c}$

total power

$$P_{M1} = \int \frac{dP_{M1}}{d\Omega}\,d\Omega = \frac{2\pi\, k^4 m^2}{2(4\pi)^2\varepsilon_0 c}\underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3}$$

$k^4 = \frac{\omega^4}{c^4}$

$$\boxed{P_{M1} = \frac{\omega^4 m^2}{4\pi\varepsilon_0\, 3c^5}}$$

compare to $P_{E1} = \frac{\omega^4 p^2}{4\pi\varepsilon_0\, 3c^3}$

$$\frac{P_{M1}}{P_{E1}} = \left(\frac{m}{cp}\right)^2 \quad \text{as } m \sim vp \Rightarrow \frac{P_{M1}}{P_{E1}} \sim \left(\frac{v}{c}\right)^2$$

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electric quadrupole radiation - radiation zone approx

$$\vec{A}_{E2} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, ik\left(\frac{i\omega}{6}\right)\hat{r}\cdot\overleftrightarrow{Q}$$

Find fields for homework!

For arbitrary charge distributions - not single frequency,

We found that for pure freq of oscilation, with electric dipole moment

$$\vec{p}(t) = \vec{p}(\omega)\, e^{-i\omega t}$$

the radiated fields in electric dipole approx are

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r},\omega)\, e^{-i\omega t} \qquad , \qquad \vec{B}(\vec{r},t) = \vec{B}(\vec{r},\omega)\, e^{-i\omega t}$$

with $\vec{E}(\vec{r},\omega) = \dfrac{k^2}{4\pi\varepsilon_0} \dfrac{e^{ikr}}{r}\, \hat{r}\times(\vec{p}(\omega)\times\hat{r})$

$$= \frac{\mu_0\omega^2}{4\pi} \frac{e^{i\omega r/c}}{r}\, \hat{r}\times(\vec{p}(\omega)\times\hat{r}) \qquad \text{using } k=\omega/c,\ c^2 = 1/\mu_0\varepsilon_0$$

$$\vec{B}(\vec{r},\omega) = -\frac{c\mu_0 k^2}{4\pi} \frac{e^{ikr}}{r}\, \vec{p}(\omega)\times\hat{r}$$

$$= -\frac{\mu_0\omega^2}{4\pi c} \frac{e^{i\omega r/c}}{r}\, \vec{p}(\omega)\times\hat{r}$$

For an arbitrary time varying charge, with electric dipole moment

$$\vec{p}(t) = \int d\omega\, \vec{p}(\omega)\, e^{-i\omega t}$$

Solutions are obtained by linear superposition

$$\vec{E}(\vec{r},t) = \int d\omega\, \vec{E}(\vec{r},\omega)\, e^{-i\omega t}$$

$$= \frac{\mu_0}{4\pi r}\, \hat{r}\times\left[\int d\omega\, e^{-i\omega(t-r/c)}\, \omega^2\, \vec{p}(\omega)\times\hat{r}\right]$$

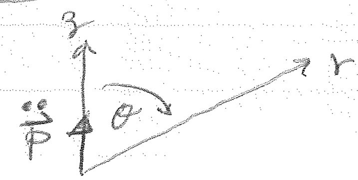
$$= \frac{\mu_0}{4\pi r}\, \hat{r}\times\left[-\frac{\partial^2}{\partial t^2} \overbrace{\int d\omega\, e^{-i\omega(t-r/c)}\, \vec{p}(\omega)}^{\vec{p}(t-r/c)} \times\hat{r}\right]$$

use triple product rule

$$\vec{E}(\vec{r},t) = \frac{-\mu_0}{4\pi r}\, \hat{r} \times \left(\ddot{\vec{p}}(t - r/c) \times \hat{r} \right)$$

second time derivative of $\vec{p}(t)$ evaluated at $\boxed{t_0 = t - r/c}$ = retarded time

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &= \frac{\mu_0}{4\pi r}\left[\left(\hat{r} \cdot \ddot{\vec{p}}(t_0) \right) \hat{r} - \ddot{\vec{p}}(t_0) \right] \\ &= \frac{\mu_0}{4\pi r}\, \ddot{p}(t_0) \sin\theta \;\hat{\theta} \end{aligned}}$$



$\hat{r} \times (\ddot{\vec{p}} \times \hat{r})$ is in $\hat{\theta}$ direction

$\ddot{p} = |\ddot{\vec{p}}|$

Similarly $\vec{B}(\vec{r},t) = \int d\omega\, \vec{B}(\vec{r},\omega)\, e^{-i\omega t}$

$$= \frac{-\mu_0}{4\pi c r} \int d\omega\, e^{-i\omega(t - r/c)} \omega^2 \vec{p}(\omega) \times \hat{r}$$

$$= \frac{\mu_0}{4\pi c r} \frac{\partial^2}{\partial t^2} \left[\int d\omega\, e^{-i\omega(t - r/c)} \vec{p}(\omega) \times \hat{r} \right]$$

$$\boxed{\begin{aligned} \vec{B}(\vec{r},t) &= \frac{\mu_0}{4\pi c r}\, \ddot{\vec{p}}(t_0) \times \hat{r} \\ &= \frac{\mu_0}{4\pi c r}\, \ddot{p}(t_0) \sin\theta \;\hat{\phi} \end{aligned}}$$

with $\ddot{\vec{p}}(t_0)$ taken along the $\hat{z}$ axis

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}\left(\frac{\mu_0}{4\pi r}\ddot{p}(t_0)\sin\theta\right)^2 \underbrace{(\hat{\theta}\times\hat{\phi})}_{=\hat{r}}$$

$$\vec{S} = \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2 \frac{\sin^2\theta}{r^2}\,\hat{r}$$

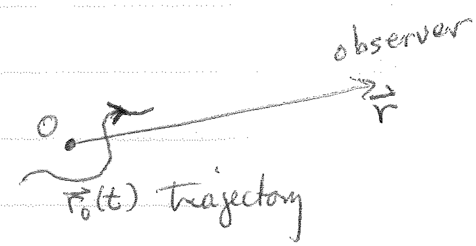
<u>total power</u> radiated through a sphere of radius r is

$$P = \oint d\vec{a}\cdot\vec{S} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, r^2 \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2 \frac{\sin^2\theta}{r^2}$$

$$= \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2 2\pi \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{\frac{4}{3}}$$

$$= \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2 \qquad \text{use } \mu_0 = \frac{1}{\varepsilon_0 c^2}$$

$$\boxed{P = \frac{1}{4\pi\varepsilon_0}\frac{2\left[\ddot{p}(t_0)\right]^2}{3c^3}}$$



For a <u>point charge moving</u> along a trajectory $\vec{r}_0(t)$

$$\vec{p}(t) = q\vec{r}_0(t) \Rightarrow \ddot{\vec{p}}(t) = q\ddot{\vec{r}}_0(t) = q\vec{a}(t)$$

↑ acceleration

$$\boxed{P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}q^2\frac{a^2(t_0)}{c^3}}$$

← total power passing through sphere of radius r at time t is due to acceleration at retarded time t_0

<u>Larmor's formular</u>

power radiated ~ (acceleration)^2

moving point charge: <u>fields</u> & <u>Poynting vector</u>

$$\vec B(\vec r,t) = \frac{\mu_0}{4\pi c r}\, q\, \vec a(t_0)\times\hat r$$

$$\boxed{\vec B(\vec r,t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{c^3}\frac{\vec a(t_0)\times\hat r}{r}}$$ using $\mu_0 = \frac{1}{\varepsilon_0 c^2}$

$$\vec E(\vec r,t) = -\frac{\mu_0 q}{4\pi r}\,\hat r\times(\vec a(t_0)\times\hat r)$$

$$\boxed{\vec E(\vec r,t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{c^2}\frac{\hat r\times(\hat r\times\vec a(t_0))}{r}}$$

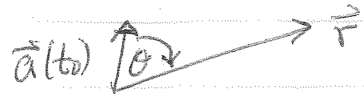
Poynting vector

$$\vec S = \frac{1}{\mu_0}\vec E\times\vec B = \frac{-1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{c^5 r^2}\underbrace{\left[\hat r\times(\hat r\times\vec a)\right]\times\left[\hat r\times\vec a\right]}_{\text{use triple product rule}}$$

$$= \frac{1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{c^5 r^2}\left\{\hat r\underbrace{(\hat r\times\vec a)^2}_{a^2-(\hat r\cdot\vec a)^2} - (\hat r\times\vec a)\underbrace{\hat r\cdot(\hat r\times\vec a)}_{=0}\right\}$$

$$\vec S = \frac{1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{c^5 r^2}\left(a^2-(\hat r\cdot\vec a)^2\right)\hat r$$

$\vec a$ evaluated at t_0



$$\vec S = \frac{1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2 a^2(t_0)}{c^5 r^2}\sin^2\theta\,\hat r$$

use $\mu_0\varepsilon_0 = \frac{1}{c^2}$

$$\boxed{\vec S = \frac{1}{4\pi\varepsilon_0}\frac{q^2 a^2(t_0)}{4\pi c^3 r^2}\sin^2\theta\,\hat r}$$