

Inertial frames of reference: Set of frames of reference which move at constant velocity with respect to each other

Special Relativity

- 1) Speed of light is constant in all inertial frames of reference
- 2) Physical laws must look the same in all inertial frames of reference - there is no experiment that can determine the "absolute" velocity of any inertial frame

⇒ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.

Equation of wavefront is $r^2 - c^2 t^2 = 0$

⇒ (x, y, z, t) coords in one inertial frame K

(x', y', z', t') coords in another inertial frame K' that moves with velocity $\vec{v} = v\hat{x}$ with respect to K .

What is the transformation that relates coords in K' to coords in K

$$y = y', \quad z = z'$$

(origins of K and K' coincide when $t = t' = 0$)

$$\Rightarrow c^2 t^2 - x^2 = c^2 t'^2 - x'^2$$

$$\Rightarrow \frac{(ct+x)(ct-x)}{(ct'+x')(ct'-x')} = 1$$

Expect transformation to be linear

$$\Rightarrow ct' + x' = (ct+x)f$$

$$ct' - x' = (ct-x)f^{-1}$$

if transf was not linear, a particle moving at constant $\vec{v}$ in one frame might look accelerated in another frame

for some constant f . Write $f = e^{-g}$ g is rel. vel

Solve for ct' and x' in terms of ct and x

$$ct' = ct \left(\frac{e^y + e^{-y}}{2} \right) - x \left(\frac{e^y - e^{-y}}{2} \right)$$

$$x' = -ct \left(\frac{e^y - e^{-y}}{2} \right) + x \left(\frac{e^y + e^{-y}}{2} \right)$$

$$ct' = ct \cosh y - x \sinh y$$

$$x' = -ct \sinh y + x \cosh y$$

meaning of parameter y

(at $x=0$)

the origin of K has trajectory $x' = -vt'$ in K'

$$\Rightarrow \frac{x'}{t'} = -v$$

from transformation above, with $x=0$, we get

$$\frac{x'}{ct'} = \frac{-ct \sinh y}{ct \cosh y} = -\tanh y$$

$$\text{so } \frac{v}{c} = \tanh y$$

$$\Rightarrow \cosh y = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\sinh y = \left(\frac{v}{c}\right) \gamma$$

Lorentz Transformation

$$\begin{cases} ct' = \gamma ct - \gamma \left(\frac{v}{c}\right) x \\ x' = -\gamma \left(\frac{v}{c}\right) ct + \gamma x \end{cases}$$

Inverse Lorentz transform obtained by taking
 $v \rightarrow -v$ in above

$$\begin{cases} ct = \gamma ct' + \gamma \left(\frac{v}{c}\right) x' \\ x = \gamma \left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}$$

Time Dilation

Consider a clock located at the origin in frame K' that moves with velocity $v\hat{x}$ as seen from "lab" frame K .

The clock in frame K' ticks at $t'_1 = 0$ and $t'_2 = T_0$.

Time between ticks in frame K' is thus T_0 .

What is time between ticks in frame K ?

Since clock is at origin of K' , then position of 1st and 2nd ticks is at $x'_1 = x'_2 = 0$.

In frame K , the observer sees tick 1 at

$$ct_1 = \gamma ct'_1 + \gamma\left(\frac{v}{c}\right)x'_1 = 0 + 0 = 0$$

and tick 2 at

$$ct_2 = \gamma ct'_2 + \gamma\left(\frac{v}{c}\right)x'_2 = \gamma c T_0 + 0 = \gamma c T_0$$

$$\text{So } t_1 = 0, \quad t_2 = \gamma T_0$$

So time between ticks as seen by K is $\Delta t = \gamma T_0 > T_0$.

So it looks to K as if K' 's clock has slowed down.

Proper Time - time between two events as measured in the frame of reference in which those two events occur at the same position.

FitzGerald Contraction

Consider frame K' moving with $v\hat{x}$ as seen by K .
A ruler, at rest in K' , has its ends located at $x'_1=0$, $x'_2=L_0$. What is the length of the ruler as seen by K ?

At $t=0$ in frame K , the observer measures the positions of the two ends of the ruler and finds

$$x'_1=0 = -\gamma\left(\frac{v}{c}\right)ct_1 + \gamma x_1 = 0 + \gamma x_1 \quad t_1=0$$
$$\Rightarrow x_1=0$$

and

$$x'_2=L_0 = -\gamma\left(\frac{v}{c}\right)ct_2 + \gamma x_2 = 0 + \gamma x_2 \quad t_2=0$$

$$\Rightarrow x_2 = \frac{L_0}{\gamma}$$

$t_1=t_2$ since observer in K measures ends of rod "at the same time"

So length of ruler in K is $x_2 - x_1 = \frac{L_0}{\gamma} < L_0$

It appears to K as if the ruler has ~~the~~ contracted.

Proper length - distance between two events as measured in the frame in which the two events happen at the same time

As seen by an observer in frame K'

Note: K 's measurement of left end occurs at time

$$ct_1' = \gamma ct_1 - \gamma \left(\frac{v}{c}\right) x_1 = 0 \Rightarrow t_1' = 0$$

K 's measurement of right end occurs at time

$$ct_2' = \gamma ct_2 - \gamma \left(\frac{v}{c}\right) x_2 = 0 - \gamma \left(\frac{v}{c}\right) \frac{L_0}{\gamma} = -\frac{v}{c} L_0$$

$$t_2' = -\frac{v}{c^2} L_0$$

So K' 's interpretation of K 's measurement is that K first measures the position of the right end of the ruler, and only a time $\frac{v}{c^2} L_0$ later measures the location of the left end.

So K' sees K measure a length

$$L_0 - \frac{v^2}{c^2} L_0$$

distance ruler travels between K 's two measurements

$$= L_0 \left(1 - \frac{v^2}{c^2}\right) = \frac{L_0}{\gamma^2}$$

So K 's two measurements, which are simultaneous to K , do not occur simultaneously to K' .

Events that are simultaneous in one frame of reference are not simultaneous in another frame of reference

So K' sees K measure a length that is according to K' a length equal to $\frac{L_0}{\gamma^2}$

But K' also sees that K is measuring with a ~~contracted~~ length scale that is ~~the~~ Fitzgerald contracted by a factor $1/\gamma$. So the length $\frac{L_0}{\gamma^2}$ seen by K' looks like the length

$\left(\frac{L_0}{\gamma^2}\right) \frac{1}{(1/\gamma)}$ when K' sees K measure it with

K 's contracted rulers. Thus K' will agree that K thinks the ruler is $\frac{L_0}{\gamma^2} \gamma = \frac{L_0}{\gamma}$ long.

K thinks the moving ruler has contracted

K' thinks K is both (i) not measuring the ends of the ruler at the same time, and (ii) measuring the length of K' 's ruler with K 's contracted ruler.

So they can both ~~see~~ agree on the outcome of what happens, but they ascribe different physical processes to what is happening.

Proper time

two events $\left\{ \begin{array}{l} (x_1, t_1) \\ (x_2, t_2) \end{array} \right\}$ seen in K

Transform to frame K' in which they are at same position $x'_1 = x'_2$. The time $t'_2 - t'_1$ in that

frame K' is the proper time between the events

$$ct'_1 = \gamma ct_1 - \gamma \left(\frac{v}{c} \right) x_1$$

$$ct'_2 = \gamma ct_2 - \gamma \left(\frac{v}{c} \right) x_2$$

$$x'_1 = -\gamma \left(\frac{v}{c} \right) ct_1 + \gamma x_1$$

$$x'_2 = -\gamma \left(\frac{v}{c} \right) ct_2 + \gamma x_2$$

$$x'_1 = x'_2 \Rightarrow \gamma(x_2 - x_1) - \gamma \left(\frac{v}{c} \right) c(t_2 - t_1) = 0$$

$$\Rightarrow \frac{x_2 - x_1}{t_2 - t_1} = v$$

So frame K' travels with $v\hat{x}$ with respect to K .

clearly can have such K' only if $v < c$.

i.e. $x_2 - x_1 < c(t_2 - t_1)$ for the two points to be timelike

Proper time

The time difference between the events in K' is

$$t'_2 - t'_1 = \gamma t_2 - \gamma \frac{v}{c^2} x_2 - \gamma t_1 + \gamma \frac{v}{c^2} x_1$$

$$= \gamma \left(t_2 - t_1 - \frac{v}{c^2} (x_2 - x_1) \right)$$

$$= \gamma \left(t_2 - t_1 - \frac{v^2}{c^2} (t_2 - t_1) \right)$$

$$= (t_2 - t_1) \gamma \left(1 - \frac{v^2}{c^2} \right) = (t_2 - t_1) \gamma / \gamma^2$$

$$\boxed{\tau \equiv t'_2 - t'_1 = \frac{t_2 - t_1}{\gamma}}$$

t_2, t_1 times in frame K
 v transforms to frame in which $x'_1 = x'_2$

Proper length

two events (x_1, t_1) (x_2, t_2) seen in K
transform to K' in which they occur at same time
 $t'_1 = t'_2$. The distance $x'_2 - x'_1$ in that frame K'
is the proper length between the two events

$$x'_1 = -\gamma\left(\frac{v}{c}\right)ct_1 + \gamma x_1$$

$$x'_2 = -\gamma\left(\frac{v}{c}\right)ct_2 + \gamma x_2$$

$$ct'_1 = \gamma ct_1 - \gamma\left(\frac{v}{c}\right)x_1$$

$$ct'_2 = \gamma ct_2 - \gamma\left(\frac{v}{c}\right)x_2$$

$$t'_1 = t'_2 \Rightarrow \gamma c(t_2 - t_1) - \gamma\left(\frac{v}{c}\right)(x_2 - x_1) = 0$$

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{c^2}{v}$$

$$\text{or } v = \frac{c^2(t_2 - t_1)}{(x_2 - x_1)}$$

such a frame K' can exist only

$$\text{if } v < c \text{ or } \frac{x_2 - x_1}{t_2 - t_1} > c$$

$x_2 - x_1 > c(t_2 - t_1)$
for the two points to
be space-like

Then the proper length is

$$l \equiv x'_2 - x'_1 = \gamma(x_2 - x_1) - \gamma\left(\frac{v}{c}\right)c(t_2 - t_1)$$

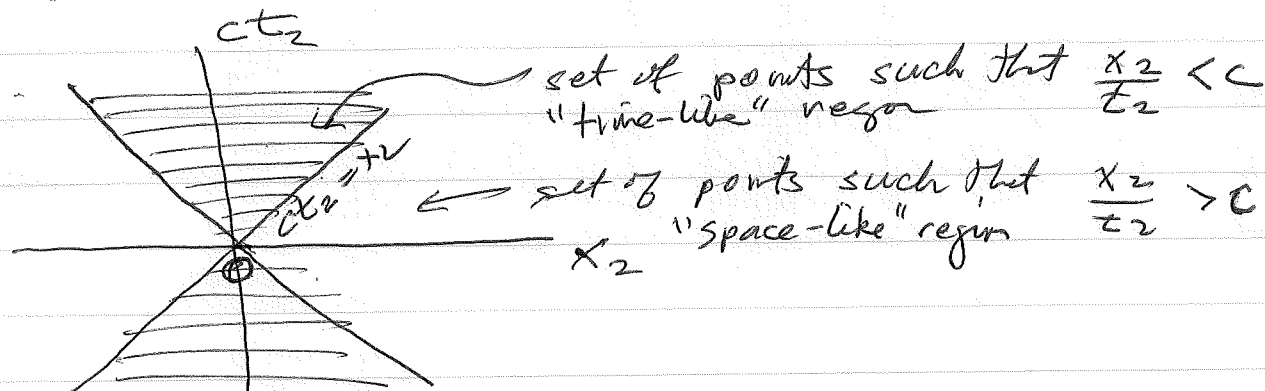
$$= \gamma(x_2 - x_1) - \gamma\left(\frac{v}{c}\right)c \frac{v}{c^2}(x_2 - x_1)$$

$$= (x_2 - x_1) \gamma \left(1 - \frac{v^2}{c^2}\right) = (x_2 - x_1) \gamma / \gamma^2$$

$$\boxed{l = \frac{x_2 - x_1}{\gamma}}$$

x_2, x_1 positions in frame K
 v transforms to frame in which $t'_1 = t'_2$

Consider two events, one of which occurs at $(x_1 = 0, t_1 = 0)$ and the other at (x_2, t_2)



The time-like region $\frac{x_2}{t_2} < c$ consists of all points such that there is a t_2 frame in which x_2 occurs at the same position as x_1 and we can therefore define the proper time between the two events.

Time-like region is such that a pulse of light emitted at origin at $t_1 = 0$ will arrive at position x_2 at a time earlier than t_2 .

The space-like region $\frac{x_2}{t_2} > c$ consists of all points such that there is a t_2 frame in which t_2 occurs at the same time as t_1 and we can therefore define the proper length between the two events.

Space-like region is such that a pulse of light emitted at origin at $t_1 = 0$ will arrive at position x_2 at a time later than t_2 .

The light cone $\frac{x_2}{t_2} = c$ separates the time-like from the space-like regions. The pt at origin can effect only events in its future time-like region. It is effected only by events in its past time-like region.