~~Inverse transform obtained by taking $v \to -v$ in above~~

$$\cancel{\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right) x' \\ x = \gamma\left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}}$$

4 - vectors

4-position: $x_\mu = (x_1, x_2, x_3, ict)$ $\qquad x_4 \equiv ict$

summation convention $x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2t^2$ Lorentz invariant scalar

- sum over repeated indicies

- has same value in all interial frames

Lorentz transf for $K \to K'$ where K' moves with $v\hat{x}$ as seen by K.

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 + i\left(\tfrac{v}{c}\right)x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\left(\tfrac{v}{c}\right)x_1\right) \end{aligned}\right\}$$

linear transf, can be represented by a matrix

or $\quad x_\mu' = a_{\mu\nu}(L)\, x_\nu$

matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $\quad x_\mu = a_{\mu\nu}(L^{-1})\, x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $\quad a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose $\Rightarrow$ "orthogonal"

More generally

Since x_μ^2 is Lorentz invariant scalar,

$$x_\mu'^2 = a_{\mu\nu}(L)\, a_{\mu\lambda}(L)\, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow \quad a_{\mu\nu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\nu\mu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\mu\nu} = a^{-1}_{\mu\nu}(L) \qquad \text{transpose} = \text{inverse}$$

a matrix whose transpose equals its inverse is called an ortogonal matrix. $a_{\mu\nu}$ is 4×4 orthogonal matrix

If L_1 is a Lorentz transf from K to K'
L_2 is a Lorentz transf from K' to K''
Then the Lorentz transf from K to K'' is given by the matrix

$$a(L_2 L_1) = a(L_2)\, a(L_1)$$

if $L_1 \equiv L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow \quad a^{-1}(L) = a(L^{-1})$$

particle on trajectory $\vec{r}(t)$
$dx_1 = x_1(t+dt) - x_1(t)$
etc

4-differential

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2 \qquad \text{Lorentz invarient scalar}$$

$$ds^2 = dt^2 \left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2\right]$$

$$ds^2 = \frac{dt^2}{\gamma^2} \qquad \vec{v} = \frac{d\vec{r}}{dt}$$

$$\boxed{ds = \frac{dt}{\gamma}}$$

proper time interval
ds is the same in all inertial frames.

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

<u>4-velocity</u> $\quad u_\mu \equiv \dfrac{dx_\mu}{ds} \equiv \dot{x}_\mu$ $\quad$ dot indicates derivative with respect to s

$$= \gamma \frac{dx_\mu}{dt}$$

since dx_μ is a 4-vector and ds is Lorentz invariant scalar, then $\dfrac{dx_\mu}{ds}$ is a 4-vector.

space components $\vec{u} = \gamma \vec{v}$

$$u_4 = ic\gamma \qquad u_\mu = \gamma(\vec{v}, ic)$$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2(v^2 - c^2)$$

$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$

Lorentz invarient scalar

<u>4-acceleration</u> $\quad a_\mu \equiv \dfrac{du_\mu}{ds} = \gamma \dfrac{du_\mu}{dt}$

<u>4-gradient</u> $\quad \dfrac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, \dfrac{-i}{c}\dfrac{\partial}{\partial t}\right) = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}, \dfrac{\partial}{\partial x_4}\right)$

where $x_4 = ict$

proof $\dfrac{\partial}{\partial x_\mu}$ is a 4-vector

by chain rule: $\quad \dfrac{\partial}{\partial x'_\mu} = \dfrac{\partial x_\lambda}{\partial x'_\mu}\dfrac{\partial}{\partial x_\lambda} \longrightarrow$ but $\dfrac{\partial x_\lambda}{\partial x'_\mu} = a_{\lambda\mu}(L^{-1}) = a_{\mu\lambda}(L)$

inverse = transpose

So $\quad \dfrac{\partial}{\partial x'_\mu} = a_{\mu\lambda}(L)\dfrac{\partial}{\partial x_\lambda}$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu}\right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

wave equation operator!

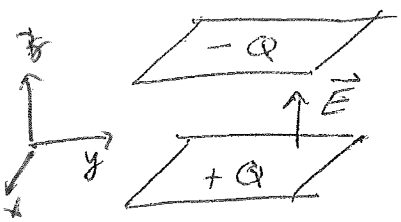
<u>inner products</u>

If U_μ and V_μ are 4-vectors, then $U_\mu V_\mu$ is Lorentz invarient scalar

<u>Maxwell's Equations in Relativistic Form</u>

How do $\vec{E}$ and $\vec{B}$ transform under Lorentz transformation?
$\vec{E}$ and $\vec{B}$ have much more complicated transformation laws than position 4-vector $X_\mu = (\vec{r}, ict)$

<u>Example</u> : parallel plate capacitor <u>at rest in K</u>
plates have area A, charge Q:



$$\vec{E} = \frac{Q}{A\varepsilon_0}\hat{z} \quad \text{uniform} \qquad \frac{Q}{A} = \sigma \text{ surface charge den.}$$

$$\vec{B} = 0$$

In K', moving with $\vec{v} = v\hat{y}$ wrt K, y dimension of plates is contracted by factor $1/\gamma$ (FitzGerald Contraction)

$$\sigma' = \frac{Q}{A'} = \frac{\gamma Q}{A} = \gamma\sigma$$

Assume Q is a Lorentz invariant scalar

$$\vec{E}' = \frac{Q}{A'\varepsilon_0}\hat{z} = \frac{\gamma Q}{A\varepsilon_0}\hat{z} = \gamma\vec{E}$$

$\vec{E}$ is along $\hat{z} \perp \vec{v}$.

This is different than transf law for $\vec{r}$.
Under L.T. components of $\vec{r} \perp \vec{v}$ <u>do not</u> change
But components of $\vec{E} \perp \vec{v}$ <u>do</u> change

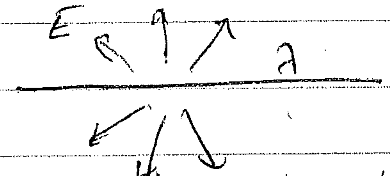
Also, moving surface charge σ' gives rise to surface current density $\Rightarrow$ there will be magnetic field $\vec{B}'$ in frame K'. $\Rightarrow$ Lorentz transf must couple together the components of $\vec{E}$ and $\vec{B}$.

Electromagnetism

Clearly $\vec{E}$ + $\vec{B}$ must transform into each other under Lorentz transf.

<u>in inertial frame K</u>

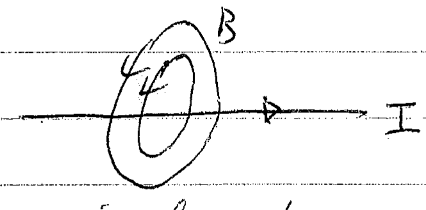
stationary line charge λ



cylindrical outward electric field

no B-field

<u>in frame K' moving with $\vec{v}$ ∥ to wire</u>



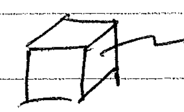
moving line charge gives current $\Rightarrow$ B circulating around wire as well as outward radial E

Lorentz force

$$\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$$

What is the velocity $\vec{v}$ here? velocity with respect to what inertial frame? Clearly $\vec{E}$ and $\vec{B}$ must change from one inertial frame to another if this force law can make sense.

charge density, current density



ΔV contains charge ΔQ

Consider charge ΔQ contained in a vol ΔV.
ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame

$\Delta Q = \overset{\circ}{\rho}\, \Delta \overset{\circ}{V}$ $\quad$ $\overset{\circ}{\rho}$ is charge density in rest frame of charge

$\Delta \overset{\circ}{V}$ is volume of box in rest frame

$\overset{\circ}{\rho}$ is a Lorentz invarient scalar by definition

Now transform to another frame moving with velocity $\vec{v}$ with respect to the rest frame.

ΔQ remains the same.

$\Delta V = \dfrac{\Delta \overset{\circ}{V}}{\gamma}$ $\quad$ volume contracts in direction $\parallel$ to $\vec{v}$

$$\Rightarrow \rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \overset{\circ}{V}}\gamma = \overset{\circ}{\rho}\gamma$$

spatial components of 4-velocity

current density is $\vec{j} = \rho\vec{v} = (\rho/\gamma)(\gamma\vec{v}) = \overset{\circ}{\rho}\,\vec{u}$

Define 4-current $j_\mu \equiv \overset{\circ}{\rho}\, u_\mu = \overset{\circ}{\rho}(\vec{u}, ic\gamma)$

spatial components of j_μ are $\vec{j} = \overset{\circ}{\rho}\vec{u} = \rho\vec{v}$ current density

temporal component of j_μ is $j_4 = ic\overset{\circ}{\rho}\gamma = ic\rho$ charge density

So $\boxed{j_\mu = (\vec{j}, ic\rho)}$ $\quad$ j_μ is a 4-vector since u_μ is a 4-vector and $\overset{\circ}{\rho}$ is Lorentz invarient scalar

length of the 4-current is $j_\mu j_\mu = |\vec{j}|^2 - c^2\rho^2 = \overset{\circ}{\rho}{}^2 u_\mu u_\mu$

$$= -c^2\overset{\circ}{\rho}{}^2$$

Charge conservation

$$0 = \vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = \vec{\nabla}\cdot\vec{j} + \frac{\partial(ic\rho)}{\partial(ict)} = \vec{\nabla}\cdot\vec{j} + \frac{\partial j_4}{\partial x_4}$$

$\Rightarrow \boxed{\dfrac{\partial j_\mu}{\partial x_\mu} = 0}$ $\quad$ charge conservation in Lorentz covariant form

Equations for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \Box^2 \vec{A} = -\mu_0 \vec{J}$$

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V = \Box^2 V = -\rho/\varepsilon_0 = -c^2 \mu_0 \rho$$

$$= -\mu_0 (ic\rho)\left(\frac{c}{i}\right)$$

So

$$\Box^2 \vec{A} = -\mu_0 \vec{J} \qquad = -\mu_0 j_4 \left(\frac{c}{i}\right)$$

$$\Box^2 (i V/c) = -\mu_0 j_4$$

Define 4-potential $\boxed{A_\mu = (\vec{A}, \, iV/c)}$

$$\Rightarrow \quad \Box^2 A_\mu = -\mu_0 j_\mu \quad \text{equation for potentials}$$

$\Box^2 = \dfrac{\partial^2}{\partial x_\nu^2}$ is Lorentz invarient operator

So we can write the above as $\boxed{\dfrac{\partial^2 A_\mu}{\partial x_\nu^2} = -\mu_0 j_\mu}$

Lorentz gauge condition is

$$0 = \vec{\nabla}\cdot\vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}$$

$$= \vec{\nabla}\cdot\vec{A} + \frac{\partial (iV/c)}{\partial (ict)} = \vec{\nabla}\cdot\vec{A} + \frac{\partial A_4}{\partial x_4}$$

$$= \frac{\partial A_\mu}{\partial x_\mu}$$

So Lorentz Gauge condition is $\boxed{\dfrac{\partial A_\mu}{\partial x_\mu} = 0}$