Electric and Magnetic Fields

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \boxed{B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}}$$

where i, j, k are a cyclic permutation of 1,2,3

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$V = \frac{cA_4}{i}, \quad x_4 = ict$$

$$\Rightarrow E_i = -\frac{\partial\left(\frac{c}{i} A_4\right)}{\partial x_i} - \frac{\partial A_i}{\partial\left(\frac{x_4}{ic}\right)} = -\frac{c}{i}\frac{\partial A_4}{\partial x_i} - ic\frac{\partial A_i}{\partial x_4}$$

$$\boxed{\frac{E_i}{c} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)}$$

has a similar form to B_i

We define the field strength tensor

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}} = -F_{\nu\mu}$$

4x4 antisymmetric 2nd rank tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4x4 antisymmetric 2nd rank tensor

4x4 antisymmetric 2nd rank tensor has only 6 independent components – just the right number to specify the $\vec{E}$ and $\vec{B}$ fields!

$F_{\mu\nu}$ transforms under a Lorentz transformation just like a tensor (ie not like a vector)

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu} \qquad \text{use} \quad \left.\begin{aligned} A'_\nu &= a_{\nu\lambda} A_\lambda \\ \frac{\partial}{\partial x'_\mu} &= a_{\mu\sigma} \frac{\partial}{\partial x_\sigma} \end{aligned}\right\} \text{since } A_\mu \text{ and } \frac{\partial}{\partial x_\sigma} \text{ are both 4-vectors}$$

$$F'_{\mu\nu} = a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma} - a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$= a_{\mu\sigma} a_{\nu\lambda} \left(\frac{\partial A_\lambda}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\lambda} \right)$$

$$\boxed{F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}} \leftarrow \text{transformation law for a 2nd rank tensor}$$

In terms of matrix multiplication, and writing for the transpose of a matrix $a_{\nu\lambda} = a^t_{\lambda\nu}$, the above can b written as

$$F'_{\mu\nu} = a_{\mu\sigma} F_{\sigma\lambda} a^t_{\lambda\nu}$$

The above has the form of the product of three matrices

If we write out the above transformation law component by component we get the following transformation law for the $\vec{E}$ and $\vec{B}$ fields.
For a transformation from K to K', where K' moves with velocity $v\hat{x}$ as seen from K,

$$E'_1 = E_1 \qquad\qquad B'_1 = B_1$$
$$E'_2 = \gamma(E_2 - vB_3) \qquad\qquad B'_2 = \gamma\left(B_2 + \frac{v}{c^2} E_3\right)$$
$$E'_3 = \gamma(E_3 + vB_2) \qquad\qquad B'_3 = \gamma\left(B_3 - \frac{v}{c^2} E_2\right)$$

where $(1,2,3) = (x,y,z)$

The transformation law for an n^{th} rank tensor is

$$T'_{\mu_1, \mu_2, \dots \mu_n} = a_{\mu_1 \nu_1} a_{\mu_2 a_2} \cdots a_{\mu_n \nu_n} T_{\nu_1, \nu_2, \dots, \nu_n}$$

Inhomogeneous Maxwell's Equations

Using the field strength tensor $F_{\mu\nu}$ we can write the inhomogeneous Maxwell's equations (ie the ones involving the sources ρ and $\vec{J}$) as follows:

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu}$$

$F_{\mu\nu}$ is a 4-tensor 2nd rank

$\frac{\partial}{\partial x_\nu}$ is a 4-vector

$\Rightarrow \frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector

Proof that $\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector. Using the transformation laws of $F_{\mu\nu}$ and $\frac{\partial}{\partial x_\nu}$ we get

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} a_{\nu\sigma} a_{\nu\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau}$$

write $\sum_\nu a_{\nu\sigma} a_{\nu\tau} = \sum_\nu a^t_{\sigma\nu} a_{\nu\tau}$

but since a is orthogonal $a^t = a^{-1}$ and $\sum_\nu a^t_{\sigma\nu} a_{\nu\tau} = \delta_{\sigma\tau}$

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} \delta_{\sigma\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau} = a_{\mu\lambda} \frac{F_{\lambda\sigma}}{\partial x_\sigma}$$

so transforms like a 4-vector

back to $\boxed{\dfrac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu}$

To see this is so, substitute in definition of $F_{\mu\nu}$ in terms of 4-potential A_μ

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

1st term = 0 by Lorentz gauge condition. So

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = -\Box^2 A_\mu = \mu_0 j_\mu$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu \Rightarrow \begin{cases} \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} & \text{spatial components} \\ \vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \rho/\varepsilon_0 & \text{temporal component} \end{cases}$$

We still need to have a Lorentz covariant way to write the homogeneous Maxwell Equations.

$\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$

<u>Homogeneous Maxwell Equations</u>

Construct the 3rd rank co-variant tensor

$$\boxed{\tilde{G}_{\mu\nu\lambda} \equiv \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}}$$

transforms as $\tilde{G}'_{\mu\nu\lambda} = a_{\mu\alpha} a_{\nu\beta} a_{\lambda\gamma} \tilde{G}_{\alpha\beta\gamma}$

$\tilde{G}_{\mu\nu\lambda}$ has in principle $4^3 = 64$ components

But can show that $\tilde{G}$ is antisymmetric in exchange of any two indicies

$$\tilde{G}_{\nu\mu\lambda} = \frac{\partial F_{\nu\mu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\lambda}}{\partial x_\nu} \qquad \text{but since } F_{\mu\nu} = -F_{\nu\mu}$$

$$= -\frac{\partial F_{\mu\nu}}{\partial x_\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x_\mu} - \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = -G_{\mu\nu\lambda}$$

$\Rightarrow$ $\tilde{G}_{\nu\mu\lambda} = 0$ if any two indicies are equal

$\Rightarrow$ there are only $\underline{\underline{4}}$ independent components of $G_{\mu\nu\lambda}$ these are

$\tilde{G}_{123}$, $\tilde{G}_{124}$, $\tilde{G}_{134}$, $\tilde{G}_{234}$

all other components are just equal to $\pm$ one of these according to permutation of indicies.

The 4 homogeneous Maxwell equations can be written as

$$\boxed{\tilde{G}_{\mu\nu\lambda} = 0}$$

To see that above is true, substitute in for $F_{\mu\nu}$ in terms of potential A_μ in definition of $\tilde{G}$

$$\tilde{G}_{\mu\nu\lambda} = \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} \underbrace{- \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda}}_{\text{cancell}} \underbrace{- \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu}}_{\text{cancel}} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}$$

cancell

Also, one has

$$\tilde{G}_{123} = \frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{23}}{\partial x_1} = \frac{\partial B_3}{\partial x_3} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_1}{\partial x_1} = 0$$

$$\tilde{G}_{123} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\tilde{G}_{412} = \frac{\partial F_{41}}{\partial x_2} + \frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_4} = \frac{i\,\partial E_1}{c\,\partial x_2} + \frac{-i\,\partial E_2}{c\,\partial x_1} + \frac{\partial B_3}{ic\,\partial t}$$

$$= \frac{i}{c}\left[\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} - \frac{\partial B_3}{\partial t}\right] = -\frac{i}{c}\left[(\vec{\nabla}\times\vec{E})_3 + \frac{\partial B_3}{\partial t}\right] = 0$$

this is the z-component of Faraday's law $\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$\tilde{G}_{413} = 0$ and $\tilde{G}_{423} = 0$ give x and y components of Faraday's law.

An alternative way to write the homogeneous Maxwell's Equations

Note: we can get the homogeneous Maxwell's equations from the inhomogeneous equations by making the substitutions

$$\vec{j} \to 0,\ \rho \to 0,\ \frac{\vec{E}}{c} \to \vec{B},\ \vec{B} \to -\frac{\vec{E}}{c}$$

So we define the <u>dual field strength tensor</u>

$$G_{\mu\nu} = \begin{pmatrix} 0 & -E_3/c & E_2/c & -iB_1 \\ E_3/c & 0 & -E_1/c & -iB_2 \\ -E_2/c & E_1/c & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$$

or equivalently if

$$\varepsilon_{\mu\nu\sigma\lambda} = \begin{cases} +1 & \text{if } \mu\nu\sigma\lambda \text{ is an even permutation of } 1234 \\ -1 & \text{if } \mu\nu\sigma\lambda \text{ is an odd permutation of } 1234 \\ 0 & \text{otherwise, ie any two indices equal} \end{cases}$$

generalization of the Levi-Civita symbol

then $\boxed{G_{\mu\nu} = \frac{1}{2i} \varepsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}}$ psuedo-tensor gives wrong sign under parity transform.

Then $\boxed{\frac{\partial G_{\mu\nu}}{\partial x_\nu} = 0}$

gives the homogeneous Maxwell's equations

From $F_{\mu\nu}$ ad $G_{\mu\nu}$ we can construct the following Lorentz invariant scalars

$$\left.\begin{aligned} \frac{1}{2} F_{\mu\nu} F_{\mu\nu} &= B^2 - \frac{E^2}{c^2} \\ -\frac{1}{4} F_{\mu\nu} G_{\mu\nu} &= \frac{\vec{B}\cdot\vec{E}}{c} \end{aligned}\right\}$$

these have the same value in any inertial frame of reference!

$\Rightarrow$ 1) If $\vec{E} \perp \vec{B}$ ad $|\vec{B}| = \frac{|\vec{E}|}{c}$ in one frame of reference, then it is so in all frames of reference. ($\vec{E}\cdot\vec{B} = 0$ ad $|\vec{B}|^2 - \frac{|\vec{E}|^2}{c^2} = 0$)

This property is satisfied by EM waves in the vacuum

2) If in one frame $\vec{E}\cdot\vec{B} = 0$ and $\frac{E^2}{c^2} > B^2$, then there exists a frame in which $\vec{B}' = 0$. If in one frame $\vec{E}\cdot\vec{B} = 0$ and $B^2 > E^2/c^2$, then there exists a frame in which $\vec{E}' = 0$.

Relativistic Kinematics

4-momentum of a particle $p_\mu = m\dot{x}_\mu = m u_\mu = (m\gamma\vec{v}, \, i m c\gamma)$

$\left\{\right.$ m is mass of particle as measured in the frame in which the particle is instantaneously at rest. m = "rest mass"

p_μ is a 4-vector since m is a scalar and u_μ is a 4-vector

$$p_\mu^2 = m^2 u_\mu^2 = -m^2c^2 \qquad \text{since } u_\mu^2 = -c^2$$

4-force $K_\mu = (\vec{K}, \, i K_0)$ also called "Minkowski force"

We guess that the relativistic generalization of Newton's 2nd law of motion is

$$\frac{m\,d^2x_\mu}{ds^2} = K_\mu \quad \text{or} \quad m\frac{du_\mu}{ds} = K_\mu$$

$$\text{or} \quad \frac{dp_\mu}{ds} = K_\mu \qquad (p_\mu = m u_\mu = m\dot{x}_\mu)$$

Now since $p_\mu^2 = -m^2c^2$ is a constant, we have

$$0 = \frac{d}{ds}(p_\mu^2) = 2p_\mu\frac{dp_\mu}{ds} = 2p_\mu K_\mu$$

$$\Rightarrow \quad p_\mu K_\mu = 0$$

$$p_\mu K_\mu = m\gamma\vec{v}\cdot\vec{K} - mc\gamma K_0 = 0$$

so $\boxed{K_0 = \frac{\vec{v}}{c}\cdot\vec{K}}$ time component of 4-force is determined by the spacial components $\vec{K}$

Define the usual 3-force by

$$\frac{d\vec{p}}{dt} = \vec{F}$$ (we identify the Newtonian momentum $\vec{p}$ with the spatial components of p_μ)

$$\frac{d\vec{p}}{ds} = \vec{K}$$ spatial part of relativistic Newton's law

$$\frac{d\vec{p}}{ds} = \gamma \frac{d\vec{p}}{dt} = \gamma \vec{F}$$ since $ds = dt/\gamma$

$$\Rightarrow \boxed{\vec{K} = \gamma \vec{F}}$$ relation between spatial part of 4-force and the usual 3-force $\vec{F}$

$$\Rightarrow K_0 = \frac{\vec{v}}{c} \cdot \vec{K} = \gamma \frac{\vec{v}}{c} \cdot \vec{F}$$

Consider now the 4-th component of Newton's equation

$$\frac{dp_4}{ds} = m \frac{du_4}{ds} = m \frac{d}{ds}(ic\gamma) = i K_0 = i \gamma \frac{\vec{v}}{c} \cdot \vec{F}$$

$$\Rightarrow \frac{d}{ds}(m\gamma c^2) = \gamma \vec{v} \cdot \vec{F}$$

$$d(m\gamma c^2) = \gamma \vec{v} \cdot \vec{F}\, ds = \gamma \vec{v} \cdot \vec{F} \frac{dt}{\gamma}$$

$$= \vec{v} \cdot \vec{F}\, dt = d\vec{r} \cdot \vec{F}$$

$\Rightarrow$

Work-energy theorem: $d(m\gamma c^2) = d\vec{r} \cdot \vec{F}$

↑ work done on particle

$\Rightarrow$ change in kinetic energy of particle

relativistic kinetic energy $\boxed{E = m\gamma c^2}$

$$p_4 = im\gamma c = iE/c$$

$$p_\mu = \left(\vec{p}, \frac{iE}{c}\right)$$ momentum-energy 4-vector

$$\vec{p} = m\gamma\vec{v}$$
$$E = m\gamma c^2$$

For particles moving at non-relativistic speeds
$v \ll c$

$$E = m\gamma c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \simeq \frac{mc^2}{1 - \frac{v^2}{2c^2}} \simeq mc^2\left(1 + \frac{v^2}{2c^2}\right)$$

$$\simeq mc^2 + \tfrac{1}{2}mv^2$$

non-relativistic kinetic energy

rest mass energy

$\frac{dp_\mu}{ds} = K_\mu$ is therefore both the relativistic analogue of Newton's 2nd law, but also the law of conservation of energy (ie the work-energy theorem)