Conservation of momentum and energy

① why is relativistic momentum ~~[illegible]~~ $\vec{p} = m\gamma\vec{v}$ and not just $m\vec{v}$ as in non-relativistic case?

Because we want momentum to be conserved in <u>all</u> frames of reference, $\vec{p}$ must be the spatial part of a 4-vector. We see this as follows.

Suppose momentum was $m\vec{v}$. For a collection of particles, conservation of momentum would mean

$$(*) \qquad \sum_i m_i \vec{v}_i(t_1) = \sum_i m_i \vec{v}_i(t_2)$$

for any times t_1 and t_2

If $(*)$ holds in one frame of reference K, and we now transform to another frame of reference K' moving with velocity $\vec{w}$ wrt K, we would find that in K', $(*)$ is no longer satisfied

$$\text{ie} \quad \sum_i m_i \vec{v}_i{}'(t_1) \neq \sum_i m_i \vec{v}_i{}'(t_2)$$

$\vec{v}_i{}'$ related to $\vec{v}_i$ and $\vec{w}$ via relativistic law for addition of velocities

see Griffiths ~~chapt 10.2.2~~ example 12.6

However, for the 4-momentum, if

$$P_\mu^{tot}(t_1) = \sum_i P_{\mu i}(t_1) = \sum_i P_{\mu i}(t_2) = P_\mu^{tot}(t_2)$$

in frame K, then $P_\mu'^{tot}(t_1) = P_\mu'^{tot}(t_2)$ in any other frame K', since $P_\mu^{tot}(t_1)$ and $P_\mu^{tot}(t_2)$ both transform the same way under Lorentz transf.

$$p_\mu^{tot}(t_1) = p_\mu^{tot}(t_2)$$

space components $\Rightarrow$ momentum conservation holds in all frames
time component $\Rightarrow$ energy conservation holds in all frames

(2) Why did we write Newton's eqn as $\frac{d\vec{p}}{dt} = \vec{F}$, with $\vec{p} = m\gamma\vec{v}$,

instead of $m\frac{d\vec{v}}{dt} = \vec{F}$ (as if used non-relativistic momentum)

If use $m\frac{d\vec{v}}{dt} = \vec{F}$, then $m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\vec{F}$

$$\frac{1}{2}m\,d(v^2) = dt\,\vec{v}\cdot\vec{F} = d\vec{r}\cdot\vec{F} = dW$$

$$\Rightarrow \frac{1}{2}m\int d(v^2) = \int dW$$

$\frac{1}{2}mv^2 = W$ get non-relativistic kinetic energy

in this formulation, energy W is not the time component of any 4-vector. Therefore if energy was conserved in one frame K, it need not be conserved in another frame K!

Only when we take $\frac{d\vec{p}}{dt} = \vec{F}$ with $\vec{p} = m\gamma\vec{v}$

do we get $\int \vec{F}\cdot d\vec{r} = m\gamma c^2 = p_0 c$ · time component of a 4-vector

$\Rightarrow$ energy conservation holds in all reference frames

Lorentz force in relativistic form

$$\frac{dp_\mu}{ds} = K_\mu$$

what is the K_μ that represents the Lorentz force? And how can we write it in a Lorentz covariant way?

K_μ should depend on the fields $F_{\mu\nu}$ and on the particle's trajectory x_μ

as $\vec{v} \to 0$ $\vec{K} = q\vec{E}$ (since magnetic force $\to 0$ as $\vec{v} \to 0$)

K_μ can't depend directly on x_μ as the force should be independent of where one puts the origin of the coordinates. So K_μ can depend only on derivatives $\dot{x}_\mu$, $\ddot{x}_\mu$, etc.

As $\vec{v} \to 0$, $\vec{K}$ does not depend on the acceleration, so $\vec{K}$ does not depend on $\ddot{x}_\mu$ or higher derivatives.

So K_μ depends only on $F_{\mu\nu}$ and $\dot{x}_\mu$

We need to form a 4-vector out of $F_{\mu\nu}$ and $\dot{x}_\mu$ that is linear in the fields $F_{\mu\nu}$ and proportional to the charge q. (since we want superposition to hold)

The only possibility is

$$K_\mu = q\, f(\dot{x}_\mu^2)\, F_{\mu\nu}\, \dot{x}_\nu$$

where $f(\mathring{x}_\mu^2)$ is some function of $\mathring{x}_\mu^2$.
But $\mathring{x}_\mu^2 = -c^2$ is a <u>constant</u>. So $f(\mathring{x}_\mu^2)$ is a constant.
That constant, $f(\mathring{x}_\mu^2)=1$, is determined by the requirement that $\vec{K} = q\vec{E}$ as $\vec{v} \to 0$.

So we have $\boxed{K_\mu = q F_{\mu\nu} \mathring{x}_\nu}$

Let's check what this gives for the ordinary 3-force
$\vec{F} = \frac{1}{\gamma}\vec{K}$

ith component $F_i = \frac{1}{\gamma} K_i = \frac{q}{\gamma}\left(\sum_{j=1}^{3} F_{ij}\mathring{x}_j + F_{i4}\mathring{x}_4\right)$

sub in for F_{ij} in terms of $\vec{A}$ $= \frac{q}{\gamma}\left(\sum_{j=1}^{3}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\mathring{x}_j - \frac{iE_i}{c}(ic\gamma)\right)$

use $\mathring{x}_4 = ic\gamma$

since $F_{i4} = \frac{-iE_i}{c}$

Now $\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \varepsilon_{ijk} B_k$

<u>proof:</u> $\varepsilon_{ijk} B_k = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial A_m}{\partial x_l}$ (using ε_{ijk} notation to take $\vec{\partial}\times\vec{A}$)

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial A_m}{\partial x_l}$$

$$= \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$$

use $\mathring{x}_j = \gamma v_j$

So $F_i = \frac{q}{\gamma}\sum_{j=1}^{3}\varepsilon_{ijk}B_k\gamma v_j + \frac{q}{\gamma}E_i\gamma$

$$= q\sum_{j=1}^{3}\varepsilon_{ijk}B_k v_j + qE_i$$

$$= qE_i + q(\vec{v}\times\vec{B})_i$$

so $\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} = \frac{1}{\gamma}\vec{K}$

The Lorentz force has the same form in all inertial frames.
No relativistic modification is needed

Relativistic Larmor's formula

non relativistic result was

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} a^2$$

total power radiated by particle with acceleration $\vec{a}$ assuming $v \ll c$

Now consider a particle moving with any speed v.
Consider the inertial frame of reference in which that particle is instantaneous at rest. Call this frame $\overset{\circ}{K}$. The velocity in this frame is thus $\overset{\circ}{\vec{v}} = 0$, and the charge is at the origin $\overset{\circ}{\vec{r}} = 0$.

The power radiated, as seen in the frame $\overset{\circ}{K}$, is then exactly

$$\overset{\circ}{P} = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \overset{\circ}{a}^2$$

where $\overset{\circ}{a}$ is the acceleration in frame $\overset{\circ}{K}$.

This result is exact because as $v/c \to 0$ all terms higher than the electric dipole term will vanish.

What we need to do is to find the way to Lorentz transform the result $\overset{\circ}{P}$ and find its value in any other frame of reference, in which the particle is moving with any velocity $\vec{v}$.

Consider the momentum-energy 4-vector describing the total momentum and total energy of the electromagnetic fields ~~of the charge~~ radiated by the charge.

in frame $\overset{\circ}{K}$ we can write this as

$$\left(\overset{\circ}{\vec{P}}_{EM}, \ \frac{i\overset{\circ}{\mathcal{E}}}{c} \right)$$

Now $\vec{P}_{EM} = \int d^3\overset{\circ}{r} \ \varepsilon_0 \overset{\circ}{\vec{E}} \times \overset{\circ}{\vec{B}}$.

But since the radiated fields are in the radial direction $\hat{\overset{\circ}{r}}$, when we integrate over all space we find $\vec{P}_{EM} = 0$.

Alternatively you have from homework, for a charge moving with small velocity $\vec{v}$, $\vec{P}_{EM} = \frac{4}{3}\frac{U}{c^2}\vec{v}$
So when $\vec{v} \to 0$, $\vec{P}_{EM} \to 0$.

So in frame $\overset{\circ}{K}$ the momentum energy 4-vector is

$$\left(0, \ \frac{i\overset{\circ}{\mathcal{E}}}{c} \right)$$

In a new frame of reference K that moves with velocity $-\vec{v}$ with respect to $\overset{\circ}{K}$ (in frame K, the charge is moving with velocity $\vec{v}$)
the energy in frame K is obtained by the transformation law for 4-vectors

$$\frac{i\mathcal{E}}{c} = \gamma \left(\frac{i\overset{\circ}{\mathcal{E}}}{c} + \frac{iv}{c} \overset{\circ}{P}_{EM\,1} \right)$$

where $\overset{\circ}{P}_{EM\,1}$ is component of $\overset{\circ}{\vec{P}}_{EM}$ in direction of $\vec{v}$. But $\overset{\circ}{P}_{EM} = 0$

So $\quad i\frac{\mathcal{E}}{c} = \gamma i \frac{\mathring{\mathcal{E}}}{c} \quad \Rightarrow \quad \mathcal{E} = \gamma \mathring{\mathcal{E}}$

similarly, if we take the origins of K and $\mathring{K}$ to coincide at the time when we are measuring the radiated power, then time transforms as

$$t = \gamma \mathring{t} + \frac{v}{c^2}\gamma \mathring{x}_1$$

where $\mathring{x}_1$ is position of charge in direction of $\vec{v}$

But charge is at origin in $\mathring{K}$ so $\mathring{x}_1 = 0$

So $\quad t = \gamma \mathring{t} \quad \leftarrow$ (since charge is not moving in $\mathring{K}$, $d\mathring{t}$ is really the proper time ds, so this is the familiar $\frac{dt}{\gamma} = ds$)

The Power radiated in frame K is then

$$\mathcal{P} = \frac{d\mathcal{E}}{dt} = \frac{\gamma d\mathring{\mathcal{E}}}{\gamma d\mathring{t}}$$

transforming $\mathcal{E} = \gamma \mathring{\mathcal{E}}$, $t = \gamma \mathring{t}$

$$= \frac{d\mathring{\mathcal{E}}}{d\mathring{t}} = \mathring{\mathcal{P}}$$

So the total radiated power is a Lorentz invariant scalar!

$$\boxed{\mathcal{P} = \mathring{\mathcal{P}} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3} q \frac{\mathring{a}^2}{c^3}}$$

where $\mathring{a}$ is acceleration of charge in its rest frame

We would like to rewrite $\mathcal{P}$ in a way that makes no explicit reference to the frame $\mathring{K}$.
ie we want to write $\mathring{a}^2$ in terms of a Lorentz invariant scalar that may be evaluated in any frame K.

Consider the 4-acceleration

$$\alpha_\mu = \frac{du_\mu}{ds} = \gamma \frac{du_\mu}{dt} \qquad \text{since } ds = dt/\gamma$$

use $u_\mu = (\gamma \vec{v}, i\gamma c)$

$$\vec{\alpha} = \gamma \frac{d}{dt}(\gamma \vec{v}) = \gamma^2 \vec{a} + \gamma \vec{v} \frac{d\gamma}{dt}$$

$$\alpha_4 = \gamma i c \frac{d\gamma}{dt}$$

we need $\dfrac{d\gamma}{dt} = \dfrac{d}{dt}\left(\dfrac{1}{\sqrt{1-v^2/c^2}}\right) = \dfrac{+\dfrac{\vec{v}}{c^2}\cdot\dfrac{d\vec{v}}{dt}}{(1-v^2/c^2)^{3/2}}$

$$= +\frac{\vec{v}\cdot\vec{a}}{c^2}\gamma^3$$

so

$$\vec{\alpha} = \gamma^2 \vec{a} + \gamma^4 \left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v}$$

$$\alpha_4 = \gamma^4 i \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)$$

$$\alpha_\mu = \gamma^4 \left(\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v} + \frac{\vec{a}}{\gamma^2} \, , \; i\left(\frac{\vec{v}\cdot\vec{a}}{c}\right) \right)$$

in frame $\overset{\circ}{K}$, $\overset{\circ}{\vec{v}} = 0$ and $\overset{\circ}{\gamma} = 1$, so

$$\overset{\circ}{\alpha}_\mu = (\overset{\circ}{\vec{a}}, 0) \quad \text{and so} \quad \overset{\circ}{a}{}^2 = \alpha_\mu^2 \quad \text{Lorentz invariant scalar}$$

So now we can write the relativistic Larmor formula

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}\overset{\circ}{a}{}^2 = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}\alpha_\mu^2 \quad \text{in any frame } K$$

In a general frame K,

$$\alpha_\mu^2 = |\vec{\alpha}|^2 + \alpha_4^2$$

$$= \gamma^8 \left[\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)^2 v^2 + \frac{a^2}{\gamma^4} + 2\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\frac{(\vec{v}\cdot\vec{a})}{\gamma^2} - \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

$$= \gamma^8 \left[-\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \left(1 - \frac{v^2}{c^2}\right) + 2\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \frac{1}{\gamma^2} + \frac{a^2}{\gamma^4} \right]$$

$$= \gamma^8 \left[\left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \left(\frac{2}{\gamma^2} - \frac{1}{\gamma^2}\right) + \frac{a^2}{\gamma^4} \right]$$

$$= \gamma^8 \left[\frac{a^2}{\gamma^4} + \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \frac{1}{\gamma^2} \right]$$

$$\alpha_\mu^2 = \gamma^4 \left[a^2 + \gamma^2 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

Note: as $v \to 0$, $\gamma \to 1$ and we get $\alpha_\mu^2 = a^2$ as we must.

So power radiated is

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 \left[a^2 + \gamma^2 \left(\frac{\vec{v}\cdot\vec{a}}{c}\right)^2 \right]$$

Examples:

① For a charge accelerating in linear motion (such as in a linear particle accelerator such as SLAC)

$\vec{v}\cdot\vec{a} = va$ since $\vec{v}$ and $\vec{a}$ are colinear

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 \left[a^2 + \gamma^2 \frac{v^2 a^2}{c^2} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q}{c^3} \gamma^4 a^2 \left[1 + \gamma^2 \frac{v^2}{c^2} \right]$$

$$1+\gamma^2\frac{v^2}{c^2} = 1 + \frac{v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2} = \gamma^2$$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}a^2\gamma^6$$

relativistic result increased by factor γ^6 compared to non-relativistic result

② For a charge accelerating in circular motion (such as in a synchrotron)

$\vec{v}\cdot\vec{a} = 0$ since $\vec{v}\perp\vec{a}$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q}{c^3}a^2\gamma^4$$

relativistic result increased by factor γ^4 compared to non-relativistic result