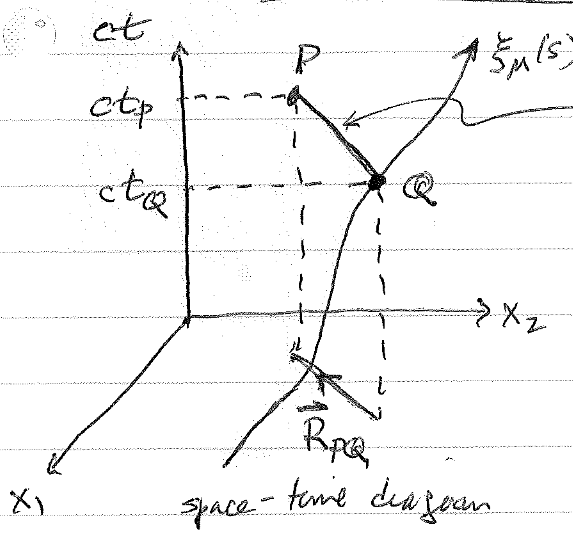
Lienard - Wiechert Potentials in Covariant form



light cone of point Q. A pulse of light leaving Q will arrive at P

space-time diagram in inertial frame K

$\xi_\mu(s)$ is the trajectory of the charged particle 4-position as function of particles proper time s

$$c\Delta t = c(t_P - t_Q) = |\vec{r}_P - \vec{r}_Q| \equiv |\vec{R}_{PQ}|$$

the 4-potential at point P is due to the charge at the earlier point Q. In frame K, $\vec{R}_{PQ}$ is the spatial vector from Q to P, Δt is the time difference between Q and P.

L-W potentials

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{q\vec{v}(t')}{|\vec{r}-\vec{r}_0(t')|} \frac{1}{1-\frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')}$$

t' is the retarded time

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r}_0(t')|} \frac{1}{1-\frac{1}{c}\hat{n}(t')\cdot\vec{v}(t')}$$

$$\hat{n}(t') = \frac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$$

if we want the potentials at point P, then $(t, \vec{r}_0(t'))$ refers to point Q. So we can rewrite the above as

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \frac{q\vec{v}_Q}{R_{PQ} - \vec{R}_{PQ}\cdot\frac{\vec{v}_Q}{c}}$$

$$\vec{r}-\vec{r}_0(t') = \vec{R}_{PQ}$$

$$V(P) = \frac{\mu_0 c^2}{4\pi} \frac{q}{R_{PQ} - \vec{R}_{PQ}\cdot\frac{\vec{v}_Q}{c}}$$

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$

re-write the denominator in a covariant form

$$R_{PQ} - \vec{R}_{PQ} \cdot \frac{\vec{v}_Q}{c} = \frac{1}{c}\left(c R_{PQ} - \vec{R}_{PQ} \cdot \vec{v} \right)$$

use $R_{PQ} = |\vec{R}_{PQ}| = c\Delta t$

$$R_{PQ} - \vec{R}_{PQ} \cdot \frac{\vec{v}_Q}{c} = \frac{1}{c}\left(c^2 \Delta t - \vec{R}_{PQ} \cdot \vec{v}_Q \right)$$

If x_μ is the 4-position of point P, and ξ_μ is the 4-position of the charge at point Q, then the 4-displacement between the two is

$$R_\mu \equiv x_\mu - \xi_\mu = (\vec{R}_{PQ}, ic\Delta t)$$

the 4-velocity of the charge at point Q is

$$u_\mu = (\gamma \vec{v}_Q, i\gamma c)$$

we then have

$$R_{PQ} - \vec{R}_{PQ} \cdot \frac{\vec{v}_Q}{c} = -\frac{1}{c\gamma} R_\mu u_\mu$$

the 4-potential is $A_\mu = \left(\vec{A}, \frac{iV}{c} \right)$ so

$$\frac{iV}{c} = \frac{i}{c} \frac{\mu_0 c^2}{4\pi} \frac{q}{R_\mu u_\mu \left(-\frac{1}{\gamma c}\right)} = -i \frac{\mu_0 c^2 \gamma}{4\pi} \frac{q}{R_\mu u_\mu}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{q \vec{v}_Q}{R_\mu u_\mu \left(-\frac{1}{\gamma c}\right)} = -\frac{\mu_0 c \gamma}{4\pi} \frac{q \vec{v}_Q}{R_\mu u_\mu}$$

So $A_\nu = \dfrac{-\mu_0 c}{4\pi} \dfrac{q}{R_\mu U_\mu} \gamma(\vec{v}_a, ic)$

$$\boxed{A_\nu = -\frac{\mu_0 c}{4\pi} \frac{q\, U_\nu}{R_\mu U_\mu}}$$ covariant form for the Lienard-Wiechert 4-potential

here $U_\mu = \dfrac{d\xi_\mu}{ds}$ is the 4-velocity of the charge at point Q.

The retarded time is determined by the condition

$R_\mu^2 = (X_\mu - \xi_\mu)^2 = 0$ holds since P is on the light cone of Q

Define the Lorentz invariant scalar

$$D \equiv -R_\mu U_\mu$$

$A_\mu(P) = \dfrac{\mu_0 c}{4\pi} \dfrac{q\, U_\mu(s)}{D}$ $U_\mu(s)$ is 4-velocity of charge at point Q.

Now we will find the fields, $F_{\mu\nu} = \dfrac{\partial A_\nu}{\partial x_\mu} - \dfrac{\partial A_\mu}{\partial x_\nu}$

When we do differentiation with respect to x_μ, we must also take care of the fact s, which locates point Q, also depends on the value of x_μ through the relation

$$R_\lambda^2 = (X_\lambda - \xi_\lambda(s))^2 = 0$$

$$\frac{\partial}{\partial x_\mu}(R_\lambda R_\lambda) = 0 \Rightarrow R_\lambda \frac{\partial R_\lambda}{\partial x_\mu} = R_\lambda \left(\delta_{\mu\lambda} - \frac{\partial \xi_\lambda}{\partial s} \frac{\partial s}{\partial x_\mu} \right) = 0$$

$$\Rightarrow R_\mu = R_\lambda U_\lambda \frac{\partial s}{\partial x_\mu} \Rightarrow \frac{\partial s}{\partial x_\mu} = \frac{R_\mu}{R_\lambda U_\lambda} = \boxed{-\frac{R_\mu}{D} = \frac{\partial s}{\partial x_\mu}}$$

Now $\frac{\partial A_\nu}{\partial x_\mu} = \frac{\mu_0 c}{4\pi} q \frac{\partial}{\partial x_\mu}\left(\frac{u_\nu}{D}\right) = \frac{\mu_0 c q}{4\pi}\left\{\frac{1}{D}\frac{\partial u_\nu}{\partial s}\frac{\partial s}{\partial x_\mu} - \frac{u_\nu}{D^2}\frac{\partial D}{\partial x_\mu}\right\}$

where $\frac{\partial D}{\partial x_\mu} = -\frac{\partial}{\partial x_\mu}(R_\lambda u_\lambda) = -R_\lambda \frac{\partial u_\lambda}{\partial s}\frac{\partial s}{\partial x_\mu} - u_\lambda \frac{\partial}{\partial x_\mu}\underbrace{(x_\lambda - \xi_\lambda)}_{R_\lambda}$

$$= -R_\lambda \dot{u}_\lambda\left(-\frac{R_\mu}{D}\right) - u_\lambda\left(\delta_{\lambda\mu} - \frac{\partial \xi_\lambda}{\partial s}\frac{\partial s}{\partial x_\mu}\right)$$

$$= \frac{R_\mu R_\lambda \dot{u}_\lambda}{D} - u_\mu + \underbrace{u_\lambda u_\lambda}_{-c^2}\left(-\frac{R_\mu}{D}\right)$$

$$\frac{\partial D}{\partial x_\mu} = -u_\mu + \frac{R_\mu c^2}{D}\left(1 + \frac{1}{c^2}\dot{u}_\lambda R_\lambda\right)$$

plug back in

$$\frac{\partial A_\nu}{\partial x_\mu} = \frac{\mu_0 c}{4\pi} q\left\{\frac{1}{D}\dot{u}_\nu\left(-\frac{R_\mu}{D}\right) - \frac{u_\nu}{D^2}\left[-u_\mu + \frac{R_\mu c^2}{D}\left(1 + \frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right]\right\}$$

$$= \frac{\mu_0 c}{4\pi} q\left\{-\frac{R_\mu \dot{u}_\nu}{D^2} + \frac{u_\nu u_\mu}{D^2} - \frac{u_\nu R_\mu c^2}{D^3}\left(1 + \frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right\}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = \frac{\mu_0 c q}{4\pi D^2}\left\{\dot{u}_\mu R_\nu - \dot{u}_\nu R_\mu + \frac{c^2}{D}\left([u_\mu R_\nu]\left[1 + \frac{\dot{u}_\lambda R_\lambda}{c^2}\right] - [u_\nu R_\mu]\left[1 + \frac{\dot{u}_\lambda R_\lambda}{c^2}\right]\right)\right\}$$

$$\boxed{F_{\mu\nu} = \frac{\mu_0 c}{4\pi}\frac{q}{D^2}\left\{\dot{u}_\mu R_\nu - \dot{u}_\nu R_\mu + \frac{c^2}{D}(u_\mu R_\nu - u_\nu R_\mu)\left(1 + \frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right\}}$$

the terms proportional to u_μ are the "velocity terms"
the terms proportional to $\dot{u}_\mu$ are the "acceleration terms"

$$F_{\mu\nu} = \frac{\mu_0 c}{4\pi} \frac{q}{D^2} \left\{ \mathring{u}_\mu R_\nu - \mathring{u}_\nu R_\mu + \frac{c^2}{D}(u_\mu R_\nu - u_\nu R_\mu)\left(1 + \frac{\mathring{u}_\lambda R_\lambda}{c^2}\right) \right\}$$

we want to show that this gives the same $\vec{E}$ as in Griffiths

$$F_{0i} = \frac{iE_i}{c} \Rightarrow E_i = -icF_{0i}$$

the pieces we need are:

4-velocity $\quad u_\mu = \gamma(\vec{v}, ic)$

4-acceleration $\quad \alpha_\mu = \mathring{u}_\mu = \left(\gamma^4 \frac{(\vec{v}\cdot\vec{a})}{c^2}\vec{v} + \gamma^2\vec{a},\ i\gamma^4\frac{(\vec{v}\cdot\vec{a})}{c} \right)$

4-difference $\quad R_\mu = (\vec{R}, iR) \quad$ displacement from Q to P

$$D = -R_\lambda u_\lambda = \gamma(cR - \vec{v}\cdot\vec{R})$$

$$\mathring{u}_\lambda R_\lambda = \gamma^4\frac{(\vec{v}\cdot\vec{a})}{c^2}(\vec{v}\cdot\vec{R}) + \gamma^2(\vec{a}\cdot\vec{R}) - \gamma^4\frac{(\vec{v}\cdot\vec{a})}{c}R$$

$$= \gamma^4\frac{(\vec{v}\cdot\vec{a})}{c^2}[\vec{v}\cdot\vec{R} - cR] + \gamma^2(\vec{a}\cdot\vec{R})$$

$$E_i = \frac{-i\mu_0 c^2}{4\pi}\frac{q}{D^3}\left\{ [\mathring{u}_0 R_i - \mathring{u}_i R_0]D + c^2(u_0 R_i - u_i R_0)\left(1 + \frac{\mathring{u}_\lambda R_\lambda}{c^2}\right) \right\}$$

use $\mu_0 c^2 = 1/\varepsilon_0$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{D^3}\left\{ [\overbrace{\gamma^4\frac{(\vec{v}\cdot\vec{a})}{c}\vec{R}}^{\mathring{u}_0 R_i} \overbrace{- \gamma^4\frac{(\vec{v}\cdot\vec{a})}{c^2}R\vec{v} - \gamma^2 R\vec{a}}^{-\mathring{u}_i R_0}]\overbrace{\gamma(cR - \vec{v}\cdot\vec{R})}^{D} \right.$$

$$\left. + \underbrace{\gamma(c\vec{R} - R\vec{v})}_{u_0 R_i - u_i R_0}\underbrace{\left(c^2 + \gamma^4\frac{(\vec{v}\cdot\vec{a})}{c^2}[\vec{v}\cdot\vec{R} - cR] + \gamma^2(\vec{a}\cdot\vec{R})\right)}_{c^2 + \mathring{u}_\lambda R_\lambda} \right\}$$

multiply through

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{D^3}\left\{ \gamma^5\frac{(\vec{v}\cdot\vec{a})}{c^2}(c\vec{R} - R\vec{v})(cR - \vec{v}\cdot\vec{R}) - \gamma^3(cR - \vec{v}\cdot\vec{R})R\vec{a} \right.$$

$$\left. + \gamma^5\frac{(\vec{v}\cdot\vec{a})}{c^2}(\vec{v}\cdot\vec{R} - cR)(c\vec{R} - R\vec{v}) + \gamma^3(c\vec{R} - R\vec{v})(\vec{a}\cdot\vec{R}) + \gamma c^2(c\vec{R} - R\vec{v}) \right\}$$

The γ^5 terms cancel

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{D^3}\left\{\gamma c^2(c\vec{R} - R\vec{v}) + \gamma^3\left[(c\vec{R} - R\vec{v})(\vec{a}\cdot\vec{R}) - \vec{a}\,(cR - \vec{v}\cdot\vec{R})R\right]\right\}$$

rewrite last term

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{D^3}\left\{\gamma c^2(c\vec{R} - R\vec{v}) + \gamma^3\left[(c\vec{R} - R\vec{v})(\vec{a}\cdot\vec{R}) - \vec{a}\,(c\vec{R} - R\vec{v})\cdot\vec{R}\right]\right\}$$

simplify the γ^3 term by using the triple product rule

$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ with $\vec{A}\equiv\vec{R}$, $\vec{B} = c\vec{R} - R\vec{v}$, $\vec{C} = \vec{a}$

substitute for D

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{\gamma^3(cR - \vec{v}\cdot\vec{R})^3}\left\{\gamma c^2(c\vec{R} - R\vec{v}) + \gamma^3\vec{R}\times\left([c\vec{R} - R\vec{v}]\times\vec{a}\right)\right\}$$

write $\begin{cases} c\vec{R} - R\vec{v} = R(c\hat{R} - \vec{v}) \\ cR - \vec{v}\cdot\vec{R} = \vec{R}\cdot(c\hat{R} - \vec{v}) \end{cases}$ since $\vec{R} = R\hat{R}$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q\,R}{[\vec{R}\cdot(c\hat{R} - \vec{v})]^3}\left\{\frac{1}{\gamma^2}c^2(c\hat{R} - \vec{v}) + \vec{R}\times\left([c\hat{R} - \vec{v}]\times\vec{a}\right)\right\}$$

use $\frac{c^2}{\gamma^2} = c^2(1 - v^2/c^2) = c^2 - v^2$

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\frac{qR}{[\vec{R}\cdot(c\hat{R} - \vec{v})]^3}\left\{(c\hat{R} - \vec{v})(c^2 - v^2) + \vec{R}\times[(c\hat{R} - \vec{v})\times\vec{a}]\right\}}$$

taking $\vec{R} = \vec{r} - \vec{r}_0(t')$, $\hat{R} = \hat{n}(t')$, $\vec{v} = \vec{v}(t')$, $\vec{a} = \vec{a}(t')$, t' the retarded time

then the above is the same as the expression written at end of lecture 21.

If one defines $\vec{w} = c\hat{R} - \vec{v}$, then

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{qR}{(\vec{R}\cdot\vec{w})^3}\left\{\vec{w}(c^2 - v^2) + \vec{R}\times(\vec{w}\times\vec{a})\right\}$$

this is the same as Griffiths Eqn (10.72) if we use his notation $\vec{R}\rightarrow\vec{r}$

Similarly we can find $\vec{B}$

$B_i = F_{jk}$ with i, j, k a cyclic permutation of $1, 2, 3$

$$B_i = \frac{\mu_0 c}{4\pi}\frac{q}{D^2}\left\{ \dot{u}_j R_k - \dot{u}_k R_j + \frac{c^2}{D}(u_j R_k - u_k R_j)\left(1+\frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right\}$$

use $\dot{u}_j R_k - \dot{u}_k R_j = (\dot{\vec{u}} \times \vec{R})_i = -(\vec{R} \times \dot{\vec{u}})_i$

$u_j R_k - u_k R_j = (\vec{u} \times \vec{R})_i = -(\vec{R} \times \vec{u})_i$

$\vec{R} = R\hat{R}$

$$\vec{B} = \frac{\hat{R}}{c} \times \left[-\frac{\mu_0 c^2}{4\pi}\frac{q}{D^2}\left\{ R\dot{\vec{u}} + \frac{c^2}{D^2} R\vec{u}\left(1+\frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right\}\right]$$

compare this to

$$\frac{\hat{R}}{c} \times \vec{E} = \frac{\hat{R}}{c} \times \left[\frac{-i\mu_0 c^2}{4\pi}\frac{q}{D^2}\left\{ \dot{u}_0 \vec{R} - \dot{\vec{u}} R_0 + \frac{c^2}{D}(u_0 \vec{R} - \vec{u} R_0)\left(1+\frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right\}\right]$$

gives zero as $\hat{R} \times \hat{R} = 0$

use $R_0 = iR$

$$\frac{\hat{R}}{c} \times \vec{E} = \frac{\hat{R}}{c} \times \left[\frac{-i\mu_0 c^2}{4\pi}\frac{q}{D^2}\left\{ -iR\dot{\vec{u}} + \frac{c^2}{D}(-iR\vec{u})\left(1+\frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right\}\right]$$

$$= \frac{\hat{R}}{c} \times \left[-\frac{\mu_0 c^2}{4\pi}\frac{q}{D^2}\left\{ R\dot{\vec{u}} + \frac{c^2}{D} R\vec{u}\left(1+\frac{\dot{u}_\lambda R_\lambda}{c^2}\right)\right\}\right]$$

$$= \vec{B}$$

so $\boxed{\vec{B} = \frac{\hat{R}}{c} \times \vec{E}}$