Magnetism in Matter

Atoms in a material can develop magnetic moments when a $\vec{B}$ is turned on.
Such moments arise from orbital motion of electrons around nucleus, and from intrinsic magnetic moments due to the intrinsic spin of electrons and nuclei.
See text for classical discussion of such effects, although really need quantum mechanics to explain properly.

Materials fall into one of three categories

1) paramagnetic: net magnet moment vanishes when $\vec{B}=0$ but when $\vec{B}\neq 0$, material develops a magnetic moment parallel to $\vec{B}$.

2) diamagnetic: net magnetic moment vanishes when $\vec{B}=0$ but when $\vec{B}\neq 0$, material develops a magnetic moment antiparallel to $\vec{B}$

3) ferromagnetic: there can be a net magnetic moment in the material even when $\vec{B}=0$

Magnetization density. When apply $\vec{B}$, develope local atomic magnetic moments

$\vec{M}$ = magnetic dipole moment per unit volume

$\int_V d^3r\, \vec{M}(\vec{r})$ = total magnetic dipole moment in volume V

Vector potential produced by magnetization $\vec{M}(r)$

in dipole approx: (see 5.79) $\quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$ dipole at origin

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{M}(\vec{r}\,') \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r'$$

$$= \frac{\mu_0}{4\pi} \int_V \vec{M}(r') \times \vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r'$$

integrate by parts
$\vec{\nabla}\times(f\vec{A}) = f\,\vec{\nabla}\times\vec{A} - \vec{A}\times\vec{\nabla}f$

$$= \frac{\mu_0}{4\pi}\left[\int_V \frac{1}{|\vec{r}-\vec{r}\,'|} \vec{\nabla}'\times\vec{M}(\vec{r}\,')\, d^3r' - \int_V \vec{\nabla}'\times\left(\frac{\vec{M}(r')}{|r-r'|}\right) d^3r'\right]$$

$$= \frac{\mu_0}{4\pi}\int \frac{1}{|\vec{r}-\vec{r}\,'|} \vec{\nabla}'\times\vec{M}(\vec{r}\,')\, d^3r' + \frac{\mu_0}{4\pi}\oint_S \frac{1}{|r-r'|} \vec{M}(r')\times d\vec{a}'$$

(see prob 1.61b)

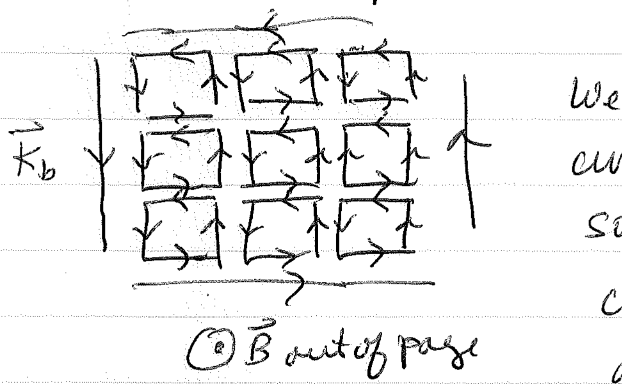
define $\quad \vec{J}_b = \vec{\nabla}\times\vec{M}$ bound current density

$\vec{K}_b \equiv \vec{M}\times\hat{n}$ bound surface current density

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_b(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r' + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{K}_b\, da'}{|\vec{r}-\vec{r}\,'|}$$

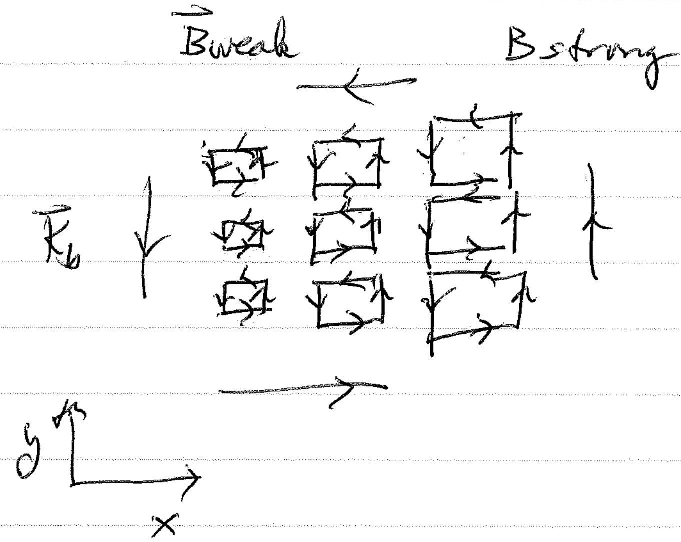
Simple picture for understanding $\vec{K}_b$ and $\vec{J}_b$

Uniform $\vec{B}$ induces uniform $\vec{M}$
model induced magnetic dipoles like small current loops



We see that inside the material, the currents from adjacent loops cancels, so $\vec{J}_b = 0$ inside. But a net surface current $\vec{K}_b$ remains. With $\vec{M} \parallel \vec{B}$ we see that the direction of $\vec{K}_b$ is indeed given by $\vec{M} \times \hat{n}$, with $\hat{n}$ the outward normal.

Non uniform $\vec{B}$ induces non-uniform $\vec{M}$



Now there is a surface current $\vec{K}_b$ as before, but also there is a $\vec{J}_b$ inside. Current on horizontal segments of adjacent loops will cancel, but current on vertical segments of adjacent loops will not cancel. There is net $\vec{J}_b$ in the $-\hat{y}$ direction

For $\vec{B} = \vec{B}(x)\hat{z}$ we expect $\vec{M} = M(x)\hat{z}$ and

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = -\frac{\partial M}{\partial x}\hat{y}$$

potential from $\vec{M}$ exact as if there were current sources $\vec{J}_b$ and $\vec{K}_b$.

Note $\vec{\nabla}\cdot\vec{J}_b = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) = 0$, as it must if $\vec{J}_b$ is to be a magnetostatic current

Above dipole approx good far away from magnetic dipoles. But we want to apply the result also inside the material, ie close to dipoles. This is same problem we had in treating electric potential V from polarization density $\vec{P}$. Similarly we find in magnetostatics, that if we ~~treat~~ consider the ~~average or~~ macroscopic $\vec{B}$ field averaged over length scales large on atomic, but small on system size, then the above dipole approx does correctly give this macroscopic B field even inside the material.

(see prob 6.11 and 5.48)

H- field

Amperes law: $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{total} = \mu_0(\vec{J}_{free} + \vec{J}_b)$

bound currents due to $\vec{M}$

"free" currents added by experimentor

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{free} + \mu_0 \vec{\nabla}\times\vec{M}$$

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_{free} \Rightarrow \vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_{free}} \Rightarrow \oint \vec{H}\cdot d\vec{l} = I_{free}^{encl}$$

ex: ferromagnet 

$\vec{\nabla}\cdot\vec{M} \neq 0$ on top and bottom surfaces

Note:

$$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = -\vec{\nabla}\cdot\vec{M} \neq 0 \text{ in general}$$

So we cannot simply replace $\vec{B}$ by $\vec{H}$ and $\vec{J}$ by $\vec{J}_{free}$ and reduce magnetostatics in matter to magnetostatics in free space. Only can do this if problem is such that $\vec{\nabla}\cdot\vec{M} = 0$, as might be the case from some symmetry.

Maxwell's eqn for magnetostatics in matter

$$\vec{\nabla}\times\vec{H} = \vec{J}_{free}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

compare to:

$$\left(\begin{array}{l}\vec{\nabla}\cdot\vec{D} = \rho_{free} \\ \vec{\nabla}\times\vec{E} = 0\end{array}\right)$$

<u>Linear magnetic media</u> (good for para or dia magnet; not true for ferromagnet)

$$\boxed{\vec{M} = \chi_m \vec{H}}$$

χ_m is magnetic susceptibility $\begin{cases}\chi_m > 0 & \text{param} \\ \chi_m < 0 & \text{diam}\end{cases}$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(\vec{H} + \chi_m\vec{H}) = \mu_0(1+\chi_m)\vec{H}$$

$$\boxed{\vec{B} = \mu\vec{H}}$$

where $\mu = \mu_0(1+\chi_m)$ is <u>permeability</u> of the material

$\vec{M} = \left(\frac{\chi_m}{\mu}\right)\vec{B}$

for linear material, $\vec{\nabla}\cdot\vec{M} = \vec{\nabla}\cdot(\chi_m\vec{H}) = \vec{\nabla}\cdot\left(\frac{\chi_m}{\mu}\vec{B}\right)$

$= \frac{\chi_m}{\mu}\vec{\nabla}\cdot\vec{B} = 0$ provided one stays <u>inside</u> material where $\frac{\chi_m}{\mu} = $ constant

but $\vec{\nabla}\cdot\vec{M} \neq 0$ at boundary between two different media, in general bound current

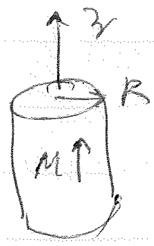
$$\vec{J}_b = \vec{\nabla} \times \vec{M} = (\vec{\nabla} \times \chi_m \vec{H}) = \chi_m \vec{\nabla} \times \vec{H} = \chi_m \vec{J}_{free}$$

$\Rightarrow$ unless free current flows through material, all bound currents are at surface.

Examples

prob 6.7

infinitely long cylinder



constant magnetization $\vec{M} = M\hat{z}$

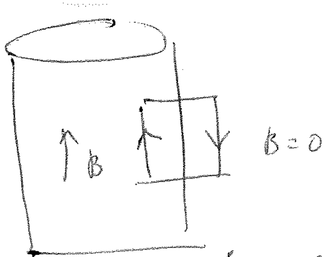
What is $\vec{B}$?

$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$ as $\vec{M}$ constant

Bound currents : $\vec{K}_b = \vec{M} \times \hat{n}$ $\quad \hat{n}$ = unit normal to surface $= \hat{r}$

$\vec{K}_b = M\hat{z} \times \hat{r} = M\hat{\phi}$ solenoidal surface current

$\Rightarrow \vec{B} = \begin{cases} 0 & \text{outside} \\ \mu_0 M \hat{z} & \text{inside} \end{cases}$



$\oint B \cdot dl = Bl = \mu_0 l K_b$

inside $\vec{B} = \mu_0 K_b \hat{z}$

here $K_b = M$

Prob
(6.12)

Infinitely long cylinder

$\vec{M} = kr\hat{z}$ frozen in magnetization

bound currents:

$\vec{J}_b = \vec{\nabla}\times\vec{M} = -\frac{\partial M_z}{\partial r}\hat{\phi}$ in cylindrical coords

$= -k\hat{\phi}$

$\vec{K}_b = \vec{M}\times\hat{n} = \vec{M}\times\hat{r} = kR\hat{\phi}$

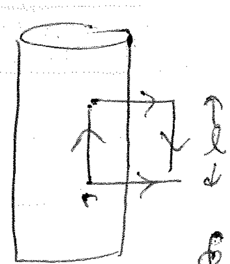
a) find $\vec{B}$ inside and outside, using the bound currents

Ampere: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

Since all currents are solenoidal, we expect $\vec{B} = B(r)\hat{z}$

also, expect $\vec{B} = 0$, $r > R$.

(View as superposition of many solenoids of radii $0 < r \leq R$)



$$\oint \vec{B}\cdot d\vec{l} = B(r)\,l = \mu_0 I_{enc}$$

$$= \mu_0 l \int_r^R dr' \underbrace{(-k)}_{\vec{J}_b(r)\cdot\hat{\phi}} + \mu_0 l \underbrace{kR}_{\vec{K}_b\cdot\hat{\phi}}$$

$$= \mu_0 l\left\{-k(R-r) + kR\right\} = \mu_0 l k r$$

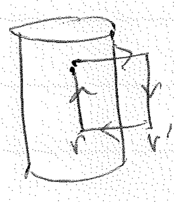
$$B(r) = \mu_0 k r$$

So $$\vec{B}(\vec{r}) = \begin{cases} \mu_0 k r \hat{z} & r \leq R \\ 0 & r > 0 \end{cases}$$

b) find $\vec{B}$ by solving first for $\vec{H}$

$$\oint \vec{H} \cdot d\vec{l} = I_{free}^{encl} = 0$$

since currents are all solenoidal, $\vec{H}$ is along $\hat{z}$ $\quad \vec{H} = H(r)\,\hat{z}$



$$\oint \vec{H} \cdot d\vec{l} = \left(H(r) - H(r')\right) l = 0$$

$\Rightarrow \vec{H}$ is constant

far from cylinder we expect $\vec{H} = 0$

$\Rightarrow \vec{H} = 0$ everywhere!

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \Rightarrow \quad \vec{B} = \mu_0 (\vec{H} + \vec{M})$$

$$= \mu_0 \vec{M} \quad \text{as } \vec{H} = 0$$

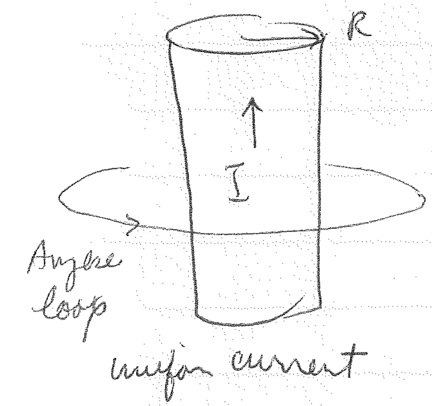
$$\vec{B} = \begin{cases} 0 & r > R \text{ outside} \\ \mu_0 k r \hat{z} & r < R \text{ inside} \end{cases}$$

same as from part (a)!

only much simpler!

prob 6.17

long straight wire, carries current I, distributed uniformly over cross sectional area of wire. Wire has χ_m. What is $\vec{B}(r)$? What are bound currents?



by symmetry, expect H is in $\hat{\phi}$ direction

$$\vec{H} = H(r)\hat{\phi}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}$$

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = I_{en}^{free}$$ take loop at radius r

$$2\pi r\, H(r) = I_{en}^{free} = \begin{cases} I & r > R \\ I\left(\frac{r^2}{R^2}\right) & r < R \end{cases}$$ as I uniform

$\left(\frac{r^2}{R^2}\right)$ ← fraction of crosssectional area enclosed by loop

$$\vec{H}(\vec{r}) = \begin{cases} \frac{I}{2\pi r}\hat{\phi} & r > R \\ \frac{I r}{2\pi R^2}\hat{\phi} & r < R \end{cases}$$

$$\vec{B} = \mu \vec{H} \qquad \text{where } \mu = \mu_0(1+\chi_m)$$

$$\Rightarrow \vec{B}(\vec{r}) = \begin{cases} \frac{\mu_0 I}{2\pi r}\hat{\phi} & r > R \quad (\chi_m = 0 \text{ outside}) \\ \frac{\mu I r}{2\pi R^2}\hat{\phi} & r < R \quad (\chi_m \neq 0 \text{ inside}) \end{cases}$$

Magnetization $\vec{M} = \chi_m \vec{H} = \begin{cases} \frac{\chi_m I r}{2\pi R^2}\hat{\phi} & r < R \\ 0 & r > R \end{cases}$

bound current density $\vec{J}_b = \vec{\nabla} \times \vec{M}$

evaluate in cylindrical coords. Since $\vec{M}$ depends only on r + points only in $\hat{\phi}$ direction

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{r}\frac{\partial}{\partial r}(rM)\,\hat{z}$$

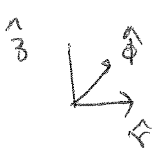
$$= \frac{1}{r}\,\frac{\chi_m I}{2\pi R^2}\,\frac{\partial}{\partial r}\left(r^2\right)\hat{z} = \frac{\chi_m I}{\pi R^2}\,\hat{z}$$

compare with $\vec{J}_{free} = \frac{I}{\pi R^2}\hat{z} \Rightarrow \vec{J}_b = \chi_m \vec{J}_f$ as expected

surface bound current

$$\vec{K}_b = \vec{M} \times \hat{n} \quad . \text{ Here } \hat{n} = \hat{r} \text{ cylindrical radial coord}$$

$$= \frac{\chi_m I R}{2\pi R^2}\,\hat{\phi} \times \hat{r} \qquad \hat{\phi} \times \hat{r} = -\hat{z}$$



$$= \frac{\chi_m I}{2\pi R}\left(-\hat{z}\right)$$

Net bound current flowing down wire is

$$\int \vec{J}_b \cdot d\vec{a} + \oint \vec{K}_b \cdot \hat{z}\, dl$$

$$= \pi R^2 \vec{J}_b \cdot \hat{z} + 2\pi R \vec{K}_b \cdot \hat{z}$$

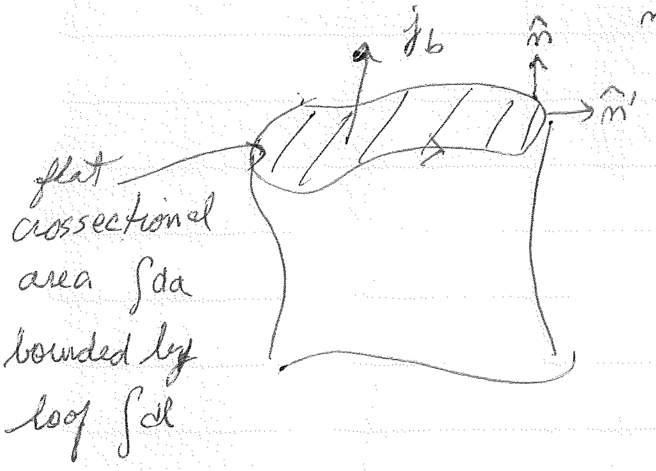
since J_b is constant

$$= \chi_m I + 2\pi R\,\frac{\chi_m I}{2\pi R}\,(-1)$$

$$= \chi_m I - \chi_m I = 0$$

<u>no net bound current flows</u>

True in general



$\hat{n} \perp$ to plane of area
$\hat{n}'$ normal to loop bounding area

$\hat{n}$ is direction down length of wire
$\hat{n}'$ is outward normal to wire

$$\int d\vec{a}\cdot\vec{J}_b + \int dl\ \vec{K}_b\cdot\hat{n}$$

$$= \int d\vec{a}\cdot(\vec{\nabla}\times\vec{M}) + \int dl\,(\vec{M}\times\hat{n}')\cdot\hat{n}$$

$$= \int d\vec{a}\cdot(\vec{\nabla}\times\vec{M}) + \int dl\,(\underbrace{\hat{n}'\times\hat{n}}_{-\hat{t}\text{ unit tangent}})\cdot\vec{M}$$

stokes ↓ , $d\vec{l} = dl\,\hat{t}$

$$= \oint d\vec{l}\cdot\vec{M} - \int d\vec{l}\cdot\vec{M} = 0$$

net bound current flowing down wire is always zero
surface current always cancells bulk current

This is because bound currents come from local circulating atomic current loops – these local atomic loops cannot lead to any net charge traveling down the wire, ie the net total bound current must sum to zero.

Equivalent to result in dielectrics that net bound charge must sum to zero since the dielectric is neutral.