decay length $= 1/k_2$ k_2 is called the attenuation

Since intensity is $\sim E^2$ decays as $e^{-2k_2 z}$, $2k_2$ is called the absorption coefficient

physical origin of decay: EM wave excits atom to oscillate. Oscillations pump energy into other degrees of freedom, due to damping γ. $\Rightarrow$ EM wave is pumping energy into material $\Rightarrow$ Energy contained in EM wave should decrease as it propagates into material $\Rightarrow$ amplitude decays.

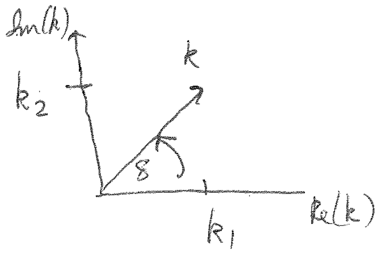
phase velocity of wave $v_p \equiv \frac{\omega}{k_1}$ depends on frequency

index of refraction $n \equiv \frac{c}{v_p} = \frac{ck_1}{\omega}$ depends on freq

Lets look now at magnetic field. From Faraday

$$\vec{B}_\omega = \frac{\vec{k}}{\omega} \times \vec{E}_\omega = \frac{(k_1 + ik_2)}{\omega} \hat{z} \times \vec{E}_\omega$$

write $k_1 + ik_2 = \sqrt{k_1^2 + k_2^2}\, e^{i\delta}$ where $\delta = \arctan\left(\frac{k_2}{k_1}\right)$
$= |k| e^{i\delta}$ is phase of k



$$\vec{B}_\omega = \frac{|k|}{\omega} \hat{z} \times \vec{E}_\omega e^{i\delta}$$

$$\vec{B}(\vec{r},t) = \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta)}$$

$$= \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z-\omega t+\delta)}$$

<u>Physical fields</u>:

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2 z}\cos(k_1 z-\omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\frac{|k|}{\omega}\, e^{-k_2 z}\cos(k_1 z-\omega t+\delta)$$

⇒ (1) $\vec{E}$ and $\vec{B}$ are transverse to $\vec{k}$, and $\vec{E}\perp\vec{B}$

(2) ratio of amplitudes $\frac{|\vec{B}|}{|\vec{E}|} = \frac{|k|}{\omega} = \frac{\sqrt{k_1^2+k_2^2}}{\omega} = \sqrt{\left|\frac{\varepsilon(\omega)}{\varepsilon_0}\right|}\,\frac{1}{c}$

(3) $\vec{B}$ wave is shifted with respect to $\vec{E}$ wave by phase shift $\delta = \arctan(k_2/k_1)$ (see Fig 8.21 in text)

<u>Summary</u>

Main consequenses of <u>complex</u> $\varepsilon(\omega)$

1) Waves decay as they propagate $\sim e^{-k_2 z}$

2) $\vec{E}$ and $\vec{B}$ waves shifted in phase by $\delta = \arctan(k_2/k_1)$

If $\varepsilon_2 = \mathrm{Im}[\varepsilon(\omega)] = 0$, then $\underbrace{\varepsilon \text{ real},}_{\text{and if } \varepsilon_1>0} \Rightarrow k$ real, $k_2 = 0$

⇒ no decay and no phase shift.

Main consequenses of <u>freq dependent</u> $\varepsilon(\omega)$

(1) $\vec{E}(t)$ and $\vec{D}(t)$ non-locally related in time

(2) waves of different ω travel with different velocities $v_p = \frac{\omega}{k_1}$

(3) <u>dispersion</u> – wave pulses do not travel with v_p, and do not keep their shape as they propagate

Phase velocity and group velocity and dispersion

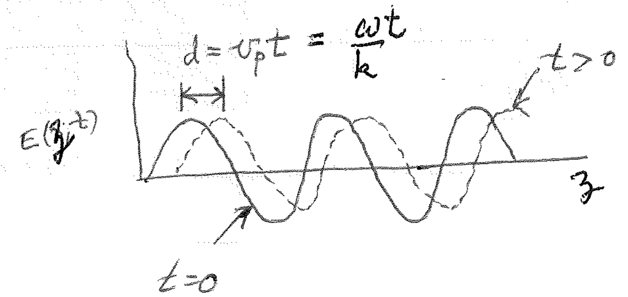
$k^2 = \frac{\omega^2}{c^2}\frac{\varepsilon(\omega)}{\varepsilon_0}$ For simplicity, assume $\varepsilon(\omega)$ is real and positive

$k = \frac{\omega}{c}\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$ $v_p = \frac{\omega}{k} = c\sqrt{\frac{\varepsilon_0}{\varepsilon(\omega)}} = \frac{c}{n}$

index of refraction $n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} = \sqrt{K(\omega)}$ dielectric function

sinusoidal waves $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ propagate with different phase speeds $v_p(\omega)$ for different ω.

v_p is speed with which peaks in oscillation move to right



for $\vec{E} = E e^{i(kz-\omega t)}$
with $\omega = v_p(\omega)\,k$

If take linear superposition of many sinusoidal waves, then each different freq ω, moves with different speed $v_p(\omega)$. So the shape of the wave is <u>not</u> preserved in time.
[This is another way to see that waves in a dielectric do not solve the wave equation – for the wave equation, all freq move with same speed v indep of ω, and the shape of the wave is always preserved in time, i.e. solutions are always of form $f(\vec{k}\cdot\vec{r} - \omega t)$]

Consider a superposition of waves all traveling in $\hat{z}$ direction

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)} \qquad k(\omega) = \frac{\omega}{c}\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

At $\vec{r}=0$, $\vec{E}(0,t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega t}$ so $\vec{E}_\omega$ is F.T. of $\vec{E}(0,t)$

At some position $\vec{r} \neq 0$

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)}$$

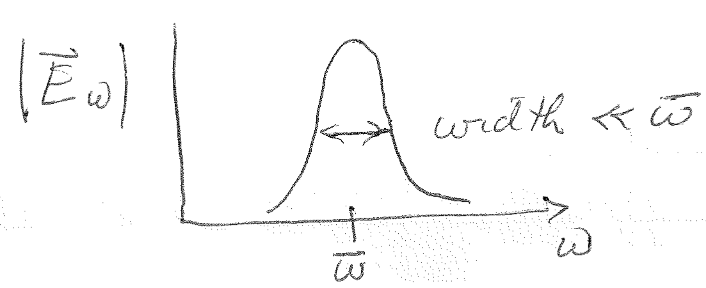
if <u>no</u> dispersion, ie $k = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}} = \frac{\omega}{v_p}$ with v_p <u>indep</u> of ω

Then $\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - z/v_p)}$

$= \vec{E}(0, t - z/v_p)$ ← form of solution to wave equation

field at z at time t, is same field as was at $z=0$ at the earlier time $t - z/v_p$ ⇒ wave moved distance z in time z/v_p ⇒ speed of wave is v_p

Suppose now that $\varepsilon(\omega)$ <u>does</u> depend on ω, so there is dispersion. Suppose $\vec{E}_\omega$ is strongly peaked about some average $\bar{\omega}$



then $k(\omega) \cong k(\bar{\omega}) + \left.\frac{dk}{d\omega}\right|_{\bar{\omega}} (\omega - \bar{\omega}) + \cdots$

$$\vec{E}(\vec{r},t) = \int d\omega\, \vec{E}_\omega\, e^{i(k(\bar{\omega})z + \frac{dk}{d\omega}\omega z - \frac{dk}{d\omega}\bar{\omega} z - \omega t)}$$

$$= e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z} \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - \frac{dk}{d\omega}z)}$$

$$= \underbrace{e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z}}_{\text{phase factor}} \;\underbrace{\vec{E}(0, t - \tfrac{dk}{d\omega}z)}_{\text{envelope - determines shape of pulse}}$$

intensity of wave $\sim |\vec{E}|^2$

$$|\vec{E}(\vec{r},t)|^2 = |\vec{E}(0, t - \tfrac{dk}{d\omega}z)|^2$$

intensity travels with velocity $v_g = 1 \Big/ \left(\frac{dk}{d\omega}\right)_{\bar{\omega}} = \left.\frac{d\omega}{dk}\right|_{\bar{\omega}} =$ group velocity

not with average phase velocity $\bar{v}_p = \frac{\bar{\omega}}{k(\bar{\omega})}$

only when $\varepsilon(\omega)$ is indep of ω will $v_p = v_g$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c} n(\omega)\right] = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\omega\frac{dn}{d\omega}} \Rightarrow$$

when $\frac{dn}{d\omega} > 0$, $v_g < v_p$ (1)

when $\frac{dn}{d\omega} < 0$, $v_g > v_p$ (2)

case (1) is called "normal" dispersion

case (2) is called "anomalous" dispersion

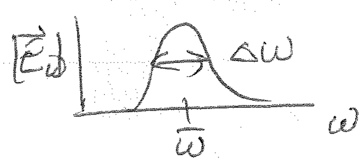
Our result $|\vec{E}(r,t)|^2 = |\vec{E}(0, t - \frac{dk}{d\omega} z)|^2$ looks like we still preserve shape of wave – but this is due to the simplicity of our approximation. If we kept to next order, i.e. used $k(\omega) = k(\bar{\omega}) + \frac{dk}{d\omega}(\omega - \bar{\omega})$

$$+ \frac{1}{2} \frac{d^2k}{d\omega^2}(\omega - \bar{\omega})^2$$

one would find that the wave pulse changes shape as it propagates – in particular, it spreads.

A simple way to estimate this effect:

If pulse initially has width $\Delta\omega$ about $\bar{\omega}$, i.e. $\vec{E}_\omega$ looks like



there is a spread in group velocities

$$\Delta v_g \simeq \left|\frac{dv_g}{d\omega}\right| \Delta\omega = \left|\frac{d}{d\omega}\left(\frac{1}{dk/d\omega}\right)\right| \Delta\omega$$

$$= \frac{1}{(dk/d\omega)^2} \left|\frac{d^2k}{d\omega^2}\right| \Delta\omega = v_g^2 \left|\frac{d^2k}{d\omega^2}\right| \Delta\omega$$

So if pulse takes a time $T = z/v_g$ to reach point z from the origin, there is also a spread in arrival times

$$\Delta T = \Delta(z/v_g) = \frac{z}{v_g^2} \Delta v_g = z \left|\frac{d^2k}{d\omega^2}\right| \Delta\omega$$

ΔT gives a spreading of width of the wave pulse, that grows linearly with the distance z traveled.

For a pulse of width $\Delta\omega$, the width in time is

$$\Delta t \sim \frac{1}{\Delta\omega} \quad \text{(like uncertainty principle in QM)}$$

$$\Rightarrow \Delta T \simeq z \left| \frac{d^2k}{d\omega^2} \right| \frac{1}{\Delta t}$$

$\Rightarrow$ the sharper the pulse is initially, (ie the smaller Δt) the faster it spreads as it travels (ie the larger ΔT is).