$\Rightarrow R_\perp = R_\parallel = 1$ This confirms that material b is totally reflecting in the region of frequency where $\varepsilon_{b1} < 0$ and $|\varepsilon_{b1}| \gg \varepsilon_{b2}$

$\varepsilon_b = \varepsilon_{b1} + i\varepsilon_{b2}$

Now consider again the special case where medium b is transparent, ie ε_b is real and $\varepsilon_b > 0$.

$$R_\perp = \left| \frac{\mu_b k_{Iz} - \mu_a k_{Tz}}{\mu_b k_{Iz} + \mu_a k_{Tz}} \right|^2 \qquad \vec{E}_0 \perp \text{plane of incidence}$$

$$R_\parallel = \left| \frac{\varepsilon_b k_{Iz} - \varepsilon_a k_{Tz}}{\varepsilon_b k_{Iz} + \varepsilon_a k_{Tz}} \right|^2 \qquad \vec{E}_0 \parallel \text{plane of incidence}$$

for a transparent medium index of refraction $n = \frac{c}{v_p} = \frac{ck}{\omega} = c\sqrt{\mu\varepsilon}$

For transparent medium b we have

$$k_{Iz} = \omega\sqrt{\mu_a \varepsilon_a}\cos\theta_I = \frac{\omega}{c} n_a \cos\theta_I$$

$$k_{Tz} = \omega\sqrt{\mu_b \varepsilon_b}\cos\theta_T = \frac{\omega}{c} n_b \cos\theta_T$$

and Snell's Law applies so $n_a \sin\theta_I = n_b \sin\theta_T$

$$\Rightarrow \frac{n_b}{n_a} = \frac{\sin\theta_I}{\sin\theta_T}$$

For simplicity we will also assume $\mu_a = \mu_b = \mu_0$

Then $$\boxed{R_\perp = \left| \frac{n_a\cos\theta_I - n_b\cos\theta_T}{n_a\cos\theta_I + n_b\cos\theta_T} \right|^2}$$

$$R_\parallel = \left| \frac{\varepsilon_b n_a \cos\theta_I - \varepsilon_a n_b \cos\theta_T}{\varepsilon_b n_a \cos\theta_I + \varepsilon_a n_b \cos\theta_T} \right|^2$$

use $\sqrt{\mu_0 \varepsilon_b} = \frac{n_b}{c} \Rightarrow \varepsilon_b = \frac{n_b^2}{c^2 \mu_0} = n_b^2 \varepsilon_0$ since $\frac{1}{c^2} = \varepsilon_0 \mu_0$

similarly $\varepsilon_a = n_a^2 \varepsilon_0$

$$\Rightarrow R_{\parallel} = \left| \frac{n_b^2 \varepsilon_0 n_a \cos\theta_I - n_a^2 \varepsilon_0 n_b \cos\theta_T}{n_b^2 \varepsilon_0 n_a \cos\theta_I + n_a^2 \varepsilon_0 n_b \cos\theta_T} \right|^2$$

$$\boxed{R_{\parallel} = \left| \frac{n_b \cos\theta_I - n_a \cos\theta_T}{n_b \cos\theta_I + n_b \cos\theta_T} \right|^2}$$

For normal incidence we have $\theta_I = \theta_T = 0$ and then we get

$$R_{\perp} = \left| \frac{n_a - n_b}{n_a + n_b} \right|^2 = R_{\parallel}$$

if $n_a = n_b$ then $R = 0$ no reflection!

$R_{\perp} = R_{\parallel}$ since for normal incidence there is no difference between the $\perp$ and $\parallel$ polarizations because $\vec{k}_I, \vec{k}_R$ and $\vec{k}_T$ are all co-linear

Consider now $\theta_I > 0$. use Snell's Law $\frac{n_b}{n_a} = \frac{\sin\theta_I}{\sin\theta_T}$

$$R_{\perp} = \left| \frac{\cos\theta_I - \frac{n_b}{n_a}\cos\theta_T}{\cos\theta_I + \frac{n_b}{n_a}\cos\theta_T} \right|^2 = \left| \frac{\cos\theta_I - \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_I}{\sin\theta_T}\right)\cos\theta_T} \right|^2$$

$$= \left| \frac{\sin\theta_T \cos\theta_I - \sin\theta_I \cos\theta_T}{\sin\theta_T \cos\theta_I + \sin\theta_I \cos\theta_T} \right|^2 = \left| \frac{\sin(\theta_I - \theta_T)}{\sin(\theta_I + \theta_T)} \right|^2$$

$$R_{\parallel} = \left(\frac{\cos\theta_I - \frac{n_a}{n_b}\cos\theta_T}{\cos\theta_I + \frac{n_a}{n_b}\cos\theta_T}\right)^2 = \left(\frac{\cos\theta_I - \left(\frac{\sin\theta_T}{\sin\theta_I}\right)\cos\theta_T}{\cos\theta_I + \left(\frac{\sin\theta_T}{\sin\theta_I}\right)\cos\theta_T}\right)^2$$

$$= \left(\frac{\sin\theta_I\cos\theta_I - \sin\theta_T\cos\theta_T}{\sin\theta_I\cos\theta_I + \sin\theta_T\cos\theta_T}\right)^2$$

$$= \left(\frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)}\right)^2$$ $\Leftarrow$ after lots of algebra

If $\theta_I + \theta_T = \frac{\pi}{2}$ then $\tan(\theta_I + \theta_T) \to \infty$ and $R_{\parallel} = 0$

This occurs at an angle of incidence $\theta_I = \theta_B$ called "Brewster's angle"

θ_B determined by Snell's law $n_a \sin\theta_I = n_b \sin\theta_T$
use $\theta_I = \theta_B$ and $\theta_T = \frac{\pi}{2} - \theta_B$

$$n_a \sin\theta_B = n_b \sin\left(\frac{\pi}{2} - \theta_B\right) = n_b \cos\theta_B$$

$$\boxed{\tan\theta_B = \frac{n_b}{n_a}}$$

For a wave incident at θ_B, the reflected wave will always have $\vec{E}_R \perp$ plane of incidence, no matter what orientation of incoming $\vec{E}_I$, since $R_{\parallel} = 0$. ~~That is only $R_{\perp} \neq 0$, so reflected wave can only have $\vec{E} \perp$ plane of incidence.~~

If incoming wave has component of $\vec{E}_I \parallel$ to plane of incidence, this component gets purely transmitted since $R_{\parallel} = 0$. Only the component of $\vec{E}_I \perp$ to plane of incidence can get reflected, since $R_{\perp} \neq 0$. $\Rightarrow$ reflected wave is <u>polorized</u> with $\vec{E}_R \perp$ to plane of incidence.

Generally, for all θ_I close to θ_B, $R_{\parallel} \ll R_{\perp}$, and the reflected wave is strongly polorized with $\vec{E}_R$ mostly $\perp$ to plane of incidence.

This is therefore one method to create a polarized light wave.

<u>Radiation from moving charges</u>

In Lorentz gauge: $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$

potentials solve $\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \equiv \Box^2 V = -\rho/\varepsilon_0$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t} \equiv \Box^2\vec{A} = -\mu_0\vec{J}$$

if we know potentials, can get fields from

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad , \quad \vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

As in electro and magneto statics, it is easier to solve for V and $\vec{A}$ and then determine $\vec{E}$ and $\vec{B}$, rather than try to solve for $\vec{E}$ and $\vec{B}$ directly.

Recall solutions for statics: $\nabla^2 V = -\rho/\varepsilon_0$, $\nabla^2\vec{A} = -\mu_0\vec{J}$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(r')}{|r-r'|} \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r')}{|r-r'|} d^3r'$$

Both solutions follow from the fact that

$$\nabla^2\left(\frac{1}{|r-r'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

↑ Dirac δ-function

$\frac{1}{|r-r'|}$ is called the "Green's function" for the operator ∇^2

Similarly, if we could find the "Green's function" for the $\Box^2$ operator, ie a function $G(\vec{r}-\vec{r}', t-t')$ that solved

$$\Box^2 G(\vec{r}-\vec{r}', t-t') = -4\pi\, \delta(\vec{r}-\vec{r}')\, \delta(t-t')$$

then the solutions to $\Box^2 V = -\rho/\varepsilon_0$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$, would be

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\; G(\vec{r}-\vec{r}', t-t')\, \rho(\vec{r}', t')$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\; G(\vec{r}-\vec{r}', t-t)\, \vec{J}(r', t')$$

How to find $G(\vec{r}-\vec{r}', t-t')$? Use Fourier transf method

$$G(\vec{r},t) = \int d^3k \int d\omega\; \tilde{G}(\vec{k},\omega)\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}$$

$$\delta(\vec{r}) = \int d^3k\, \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3} \qquad \text{from HW}$$

$$\delta(t) = \int d\omega\, \frac{e^{-i\omega t}}{2\pi}$$

substitute into $\Box^2 G(\vec{r},t) = -4\pi\, \delta(\vec{r})\, \delta(t)$

$$\int d^3k \int d\omega\; \tilde{G}(k,\omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t} = -4\pi \int d^3k \int d\omega\, \frac{e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}}{(2\pi)^4}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\int d^3k \int d\omega \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \left(\frac{\omega^2}{c^2}-k^2\right)\tilde{G}(k,\omega) = \int d^3k \int d\omega \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \frac{(-4\pi)}{(2\pi)^4}$$

equate Fourier coefficients

$$\Rightarrow \left(\frac{\omega^2}{c^2}-k^2\right)\tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4}$$

$$\tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4}\frac{c^2}{(\omega^2-c^2k^2)}$$

$$G(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tilde{G}(k,\omega)$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} \frac{d\omega \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{(2\pi)^4} \frac{(-)4\pi c^2}{(\omega+ck)(\omega-ck)}$$

integrand diverges when $\omega = \pm ck$

Can evaluate using methods of complex contour integration (see complex variables course)

$$G(\vec{r},t) = \begin{cases} 0 & t<0 \\ \frac{c}{r}\,\delta(r-ct) & t>0 \end{cases} \quad \text{where } r = |\vec{r}|$$

$$\boxed{G(\vec{r},t) = \begin{cases} 0 & t<0 \\ \frac{1}{r}\,\delta\left(t-\frac{r}{c}\right) & t>0 \end{cases}}$$

as $\delta(ax) = \frac{\delta(x)}{a}$

$G(\vec{r},t)$ has a reasonable form:

1) $G(\vec{r},t) = 0$ for $t<0$ $\Rightarrow$ no response to things in the future

2) $G(\vec{r},t) \sim \delta(|\vec{r}|-ct)$ $\Rightarrow$ response travels with speed c
response from source at $\vec{r}'=0$, $t'=0$,
is only felt at ~~time $t=r$~~ position $\vec{r}$
at time $t=\frac{r}{c}$ later.

3) If take $c\to\infty$, $G(\vec{r},t) \to \frac{\delta(t)}{r}$ response instantaneous
and $\frac{1}{r}$ is Greens function of ∇^2

expected as $\lim_{c\to\infty} \Box^2 = \nabla^2$

Explicit check that $G = \frac{c}{r}\delta(r-ct)$ solves $\Box^2 G = -4\pi\,\delta(\vec{r})\,\delta(t)$

$$\nabla^2(ab) = a\nabla^2 b + b\nabla^2 a + 2\vec{\nabla}a\cdot\vec{\nabla}b$$

$$\nabla^2\left[\frac{c}{r}\delta(r-ct)\right] = \frac{c}{r}\nabla^2\delta(r-ct) + 2\,\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) + \delta(r-ct)\,\nabla^2\left(\frac{c}{r}\right)$$

use $\nabla^2\delta(r-ct) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\delta(r-ct)\right)$ in spherical coords

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\delta'(r-ct)\right) = \frac{1}{r^2}\left[2r\,\delta'(r-ct) + r^2\delta''(r-ct)\right]$$

$$= \frac{2}{r}\delta'(r-ct) + \delta''(r-ct)$$

here $\delta'(x) = \frac{d\delta(x)}{dx}$ etc.

$$\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) = \left(\frac{-c}{r^2}\right)\left(\delta'(r-ct)\right)$$

(evaluated in spherical coords)

$$\nabla^2\left(\frac{c}{r}\right) = -4\pi\,\delta(\vec{r})\,c$$

so $\nabla^2\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{c}{r}\left(\frac{2}{r}\delta'(r-ct) + \delta''(r-ct)\right)$

$$- \frac{2c}{r^2}\delta'(r-ct) \quad - 4\pi c\,\delta(\vec{r})\,\delta(r-ct)$$

$$= \frac{c}{r}\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta\left(t-\frac{r}{c}\right)$$ using $\delta(r-ct) = \frac{\delta(t-\frac{r}{c})}{c}$

$$= \frac{c}{r}\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta(t)$$

↑ since $r=0$ because of $\delta(r)$ term

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{c}{r}(-c)\,\delta'(r-ct)\right]$$

$$= -\frac{1}{r}\frac{\partial}{\partial t}\delta'(r-ct) \;=\; \frac{c}{r}\delta''(r-ct)$$

so $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{c}{r}\,\delta(r-ct)\right) = \frac{c}{r}\delta''(r-ct) - 4\pi\,\delta(\vec{r})\,\delta(t)$

$$-\frac{c}{r}\delta''(r-ct)$$

$$= -4\pi\,\delta(\vec{r})\,\delta(t)$$ as desired

$$V(\vec{r},t) = \int \frac{d^3r'}{4\pi\varepsilon_0}\;\frac{\rho(r',t')}{|\vec{r}-\vec{r}'|}$$

where $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$ "retarded time"

depends on $\vec{r}$ and $\vec{r}'$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\;\frac{\vec{J}\left(\vec{r}',\,t'=t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

general solution to inhomogeneous wave equation

Summary

The Green's function solves

$$\Box^2 G(\vec{r}-\vec{r}\,', t-t') = -4\pi\,\delta(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

Therefore the solution to

$$\Box^2 V(\vec{r},t) = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

is

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\; \rho(\vec{r}\,',t')\, G(\vec{r}-\vec{r}\,', t-t')$$

check:

$$\Box^2 V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\; \rho(\vec{r}\,',t')\, \Box^2 G(\vec{r}-\vec{r}\,', t-t')$$

since $\Box^2$ acts on $\vec{r}$ and t not on $\vec{r}\,'$ and t'

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\; \rho(\vec{r}\,',t')\,(-4\pi)\,\delta(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

integrate over the δ-functions

$$= -\frac{\rho(r,t)}{\varepsilon_0}$$

Similarly, the solution to $\Box^2 \vec{A}(\vec{r},t) = \mu_0 \vec{j}(\vec{r},t)$ is

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\; \vec{j}(\vec{r}\,',t')\, G(\vec{r}-\vec{r}\,', t-t')$$

Now our solution for the Greens function was

$$G(\vec{r}-\vec{r}\,', t-t') = \begin{cases} 0 & t < t' \\ \dfrac{1}{|\vec{r}-\vec{r}\,'|}\,\delta\!\left(t-t'-\dfrac{|\vec{r}-\vec{r}\,'|}{c}\right) & t > t' \end{cases}$$

So we can write our solution for $V(\vec{r},t)$ as

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{t} dt' \, \rho(\vec{r}',t') \frac{1}{|\vec{r}-\vec{r}'|} \, \delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)$$

do integral over the δ-function

$$= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} d^3r' \, \frac{\rho\left(\vec{r}', \, t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

or $$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} d^3r' \, \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}$$

where here t' is the "retarded" time

$$t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$$ depends on $\vec{r}'$

Similarly
$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} d^3r' \, \frac{\vec{j}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}$$

$$t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$$