The multipole expansion in electro- and magneto-statics

Electrostatics

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2} = r\sqrt{1 - 2\frac{\hat{r}\cdot\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2}$$

expand to 2nd order in $\left(\frac{r'}{r}\right)$ $\qquad \sqrt{1+\delta} = 1 + \frac{\delta}{2} - \frac{\delta^2}{8}$

$$|\vec{r}-\vec{r}'| \simeq r\left[1 - \frac{\hat{r}\cdot\vec{r}'}{r} + \frac{1}{2}\left(\frac{r'}{r}\right)^2 - \frac{(\hat{r}\cdot\vec{r}')^2}{2r^2}\right]$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\,\frac{1}{1 - \frac{\hat{r}\cdot\vec{r}'}{r} + \frac{1}{2r^2}\left(r'^2 - (\hat{r}\cdot\vec{r}')^2\right)}$$

expand to 2nd order in $\left(\frac{r'}{r}\right)$ $\qquad \frac{1}{1+\delta} = 1 - \delta + \delta^2$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\left[1 + \frac{\hat{r}\cdot\vec{r}'}{r} - \frac{1}{2r^2}\left(r'^2 - (\hat{r}\cdot\vec{r}')^2\right) + \left(\frac{\hat{r}\cdot\vec{r}'}{r}\right)^2\right]$$

$$= \frac{1}{r}\left[1 + \frac{\hat{r}\cdot\vec{r}'}{r} + \frac{1}{2r^2}\left(3(\hat{r}\cdot\vec{r}')^2 - r'^2\right)\right]$$

$$= \frac{1}{r}\left[1 + \frac{\hat{r}\cdot\vec{r}'}{r} + \frac{1}{2r^2}\,\hat{r}\cdot\left(3\vec{r}'\vec{r}' - r'^2\overleftrightarrow{I}\right)\cdot\hat{r}\right]$$

tensor $(\vec{r}'\vec{r}')_{ij} = r'_i r'_j$ $\qquad$ identity tensor $I_{ij} = \delta_{ij}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r}\int d^3r'\,\rho(\vec{r}') + \frac{\hat{r}\cdot}{4\pi\varepsilon_0 r^2}\left[\int d^3r'\,\vec{r}'\rho(\vec{r}')\right]$$

$$+ \frac{1}{4\pi\varepsilon_0 r^3}\,\frac{1}{2}\,\hat{r}\cdot\left[\int d^3r'\left(3\vec{r}'\vec{r}' - r'^2\overleftrightarrow{I}\right)\rho(\vec{r}')\right]\cdot\hat{r} + \dots$$

Define: $q = \int d^3r\, \rho(\vec{r})$ monopole moment
0th moment = total charge

1st moment $\vec{p} = \int d^3r\, \vec{r}\rho(\vec{r})$ electric dipole moment

$\Rightarrow p_i = \int d^3r\, r_i \rho(\vec{r})$ $\vec{p} = \sum_i q_i \vec{r}_i$ for point charges

2nd moment $\overleftrightarrow{Q} = \int d^3r\, (3\vec{r}\vec{r} - r^2 \overleftrightarrow{I})\rho(\vec{r})$ electric quadrupole tensor

$$\Rightarrow Q_{ij} = \int d^3r\, (3r_i r_j - r^2 \delta_{ij})\rho(\vec{r})$$

$Q_{ij} = Q_{ji}$ is symmetric

$$\mathrm{Trace}[\overleftrightarrow{Q}] = \sum_i Q_{ii} = \sum_i \int d^3r\, (3r_i^2 - r^2)\rho(\vec{r}) = 0$$

since $\sum_i 3r_i^2 = 3r^2$, $\sum_i r^2 = 3r^2$

$$V(\vec{r}) \simeq \underbrace{\frac{q}{4\pi\varepsilon_0 r}}_{\text{monopole}} + \underbrace{\frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^2}}_{\text{dipole}} + \underbrace{\frac{\frac{1}{2}\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{4\pi\varepsilon_0 r^3}}_{\text{quadrupole}} + \cdots$$

Magnetostatics

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

expand $\frac{1}{|\vec{r}-\vec{r}'|}$ as before

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi r} \int d^3r' \, \vec{J}(\vec{r}') + \frac{\mu_0}{4\pi r^2} \int d^3r' (\hat{r}\cdot\vec{r}')\vec{J}(\vec{r}') + \cdots$$

$$= \frac{\mu_0}{4\pi r} \vec{I}_1 + \frac{\mu_0}{4\pi r^2} \vec{I}_2$$

with $\vec{I}_1$ and $\vec{I}_2$ the same as we saw before in our expansion for the solution of $\vec{A}$ to the inhomogeneous wave equation

there we found that $\vec{I}_1 = \int d^3r' \, \vec{J}(\vec{r}') = -\int d^3r' \, \vec{r}' (\vec{\nabla}' \cdot \vec{J})$

but in magnetostatics we must have $\vec{\nabla} \cdot \vec{J} = 0$ since $\frac{\partial \rho}{\partial t} = 0$
so in magnetostatics, $\vec{I}_1 = 0$. There is no magnetic monopole term.

And we found $\vec{I}_2 = \int d^3r' (\hat{r}\cdot\vec{r}')\vec{J}(\vec{r}') = -\hat{r}\times\vec{m} - \frac{1}{2}\frac{i\omega}{3}\hat{r}\cdot\overleftrightarrow{Q}'$

but in statics $\omega = 0$ so $\vec{I}_2 = -\hat{r}\times\vec{m}$
where $\vec{m} = \frac{1}{2}\int d^3\vec{r}\times\vec{J}$ is the static magnetic dipole moment

$$\vec{A}(\vec{r}) \simeq \frac{-\mu_0}{4\pi r^2}\hat{r}\times\vec{m} = \frac{\mu_0}{4\pi r^2}\vec{m}\times\hat{r}$$

gives the vector potential in the magnetic dipole approximation

Dipole fields in statics

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{\hat{r}\cdot\vec{p}}{r^2} = \frac{1}{4\pi\varepsilon_0}\,\frac{\vec{r}\cdot\vec{p}}{r^3}$$

$$\vec{E} = -\vec{\nabla}V = -\frac{1}{4\pi\varepsilon_0}\,\vec{\nabla}\left(\frac{\vec{r}\cdot\vec{p}}{r^3}\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{3}\hat{e}_i\,\frac{\partial}{\partial r_i}\left(\sum_{k=1}^{3}\frac{r_k p_k}{r^3}\right)$$

$\hat{e}_i$ is unit vector in direction i

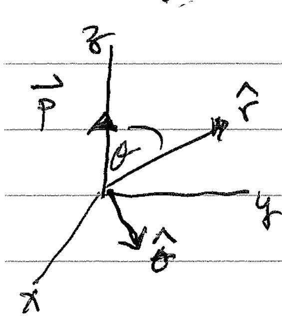
$$= -\frac{1}{4\pi\varepsilon_0}\sum_{i,k=1}^{3}\hat{e}_i\left[\frac{\delta_{ik}p_k}{r^3} - \frac{3 r_k p_k r_i}{r^5}\right]$$

where we used $\frac{\partial}{\partial r_i}\left(\frac{1}{r^3}\right) = \frac{\partial}{\partial r_i}\left[\sum_{\ell=1}^{3} r_\ell^2\right]^{-3/2} = -\frac{3}{2}\left[\sum_\ell r_\ell^2\right]^{-5/2} 2r_i$

$$= -\frac{3r_i}{r^5}$$

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}}{r^3} - \frac{3(\vec{r}\cdot\vec{p})\vec{r}}{r^5}\right] = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

in spherical coordinates: choose z axis to align with $\vec{p}$



$\vec{p}\cdot\hat{r} = p\cos\theta$ $\qquad\qquad$ $\vec{p} = p\hat{z}$

$$\vec{p} = p\cos\theta\,\hat{r} - p\sin\theta\,\hat{\theta}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3p\cos\theta\,\hat{r} - p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}\right]$$

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$$\vec{A} = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\frac{\vec{m} \times \vec{r}}{r^3} \right)$$

Levi-Civita symbol $\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is even permutation of } 123 \\ -1 & \text{if } ijk \text{ is odd permutation of } 123 \\ 0 & \text{otherwise, in particular if any two of the } ijk \text{ are equal} \end{cases}$

If $\vec{C} = \vec{A} \times \vec{B}$ then $C_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_j B_k$

example $C_x = \sum_{jk} \varepsilon_{xjk} A_j B_k = A_y B_z - A_z B_y$

Also

$$\sum_{k=1}^{3} \varepsilon_{kij} \varepsilon_{klm} = \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right)$$

Then

$$B_i = \frac{\mu_0}{4\pi} \sum_{jk} \varepsilon_{ijk} \frac{\partial}{\partial r_j} \left[\sum_{lm} \varepsilon_{klm} \frac{m_l r_m}{r^3} \right] \qquad [\cdots] = \left(\frac{\vec{m} \times \vec{r}}{r^3} \right)_k$$

$$= \frac{\mu_0}{4\pi} \sum_{jklm} \varepsilon_{kij} \varepsilon_{klm} m_l \frac{\partial}{\partial r_j} \left(\frac{r_m}{r^3} \right)$$

used $\varepsilon_{ijk} = \varepsilon_{kij}$ even permutation

$$= \frac{\mu_0}{4\pi} \sum_{jlm} \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) m_l \left[\frac{\delta_{jm}}{r^3} - \frac{3 r_m r_j}{r^5} \right]$$

$$= \frac{\mu_0}{4\pi} \left[\frac{3 m_i}{r^3} - \frac{3 m_i (\vec{r} \cdot \vec{r})}{r^5} - \frac{m_i}{r^3} + \frac{3 r_i (\vec{r} \cdot \vec{m})}{r^5} \right]$$

use $\sum_{jlm} \delta_{il}\,\delta_{jm}\, m_l \frac{\delta_{jm}}{r^3} = \frac{m_i}{r^3} \sum_{jm} \delta_{jm} = \frac{m_i}{r^3} \sum_j (1) = \frac{3m_i}{r^3}$

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left[3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}\right]$$

this has the same form as the dipolar $\vec{E}$ field except with $\vec{p} \to \vec{m}$

So in spherical coordinates with $\hat{z}$ aligned with $\vec{m}$

$\vec{m} = m\hat{z}$

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

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magnetic dipole moment:

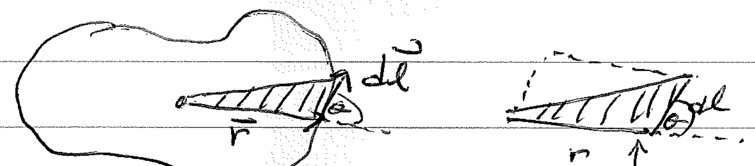
$\vec{m} = \frac{1}{2}\int d^3r\,(\vec{r}\times\vec{J})$ in general

for a wire loop

$$\vec{m} = \frac{1}{2}\oint_{loop} dl\,(\vec{r}\times\vec{I}) = \frac{I}{2}\oint_{loop} \vec{r}\times d\vec{l}$$

since I is constant in loop and points in tangential direction $d\vec{l}$

For a planar loop – a loop that lies flat in a plane



area is $(r\,dl\sin\theta)\frac{1}{2}$

$\vec{r}\times d\vec{l} = r\,dl\sin\theta\;\hat{n}$

$\hat{n}$ normal to plane

$$\vec{m} = \hat{n}\,I\int dl\,\frac{r\sin\theta}{2} = \hat{n}\,I\,(\text{area of loop})$$