For arbitrary charge distributions - not single frequency

We found that for pure freq of oscilation, with $\overbrace{\text{dipole}}^{\text{electric}}$ moment

$$\vec{p}(t) = \vec{p}(\omega)\, e^{-i\omega t}$$

the radiated fields in electric dipole approx are

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r},\omega)\, e^{-i\omega t} \qquad , \qquad \vec{B}(\vec{r},t) = \vec{B}(\vec{r},\omega)\, e^{-i\omega t}$$

with $\vec{E}(\vec{r},\omega) = \dfrac{k^2}{4\pi\varepsilon_0}\,\dfrac{e^{ikr}}{r}\,\hat{r}\times(\vec{p}(\omega)\times\hat{r})$

$$= \frac{\mu_0\omega^2}{4\pi}\,\frac{e^{i\omega r/c}}{r}\,\hat{r}\times(\vec{p}(\omega)\times\hat{r}) \qquad \text{using } k = \omega/c, \quad c^2 = 1/\mu_0\varepsilon_0$$

$$\vec{B}(\vec{r},\omega) = -\frac{c\mu_0 k^2}{4\pi}\,\frac{e^{ikr}}{r}\,\vec{p}(\omega)\times\hat{r}$$

$$= -\frac{\mu_0\omega^2}{4\pi c}\,\frac{e^{i\omega r/c}}{r}\,\vec{p}(\omega)\times\hat{r}$$

For an arbitrary time varying charge, with $\overbrace{\text{dipole}}^{\text{electric}}$ moment

$$\vec{p}(t) = \int d\omega\, \vec{p}(\omega)\, e^{-i\omega t}$$

Solutions are obtained by linear superposition

$$\vec{E}(\vec{r},t) = \int d\omega\, \vec{E}(\vec{r},\omega)\, e^{-i\omega t}$$

$$= \frac{\mu_0}{4\pi r}\,\hat{r}\times\left[\int d\omega\, e^{-i\omega(t-r/c)}\,\omega^2\,\vec{p}(\omega)\times\hat{r}\right]$$

$$= \frac{\mu_0}{4\pi r}\,\hat{r}\times\left[-\frac{\partial^2}{\partial t^2}\overbrace{\int d\omega\, e^{-i\omega(t-r/c)}\,\vec{p}(\omega)}^{\vec{p}(t-r/c)}\times\hat{r}\right]$$

use triple product rule

$$\vec{E}(\vec{r},t) = \frac{-\mu_0}{4\pi r}\,\hat{r}\times\left(\ddot{\vec{p}}(t-r/c)\times\hat{r}\right)$$

second time derivative of $\vec{p}(t)$ evaluated at $\boxed{t_0 = t - r/c}$ = retarded time

$$\boxed{\begin{aligned}\vec{E}(\vec{r},t) &= \frac{\mu_0}{4\pi r}\left[\left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right)\hat{r} - \ddot{\vec{p}}(t_0)\right] \\ &= \frac{\mu_0}{4\pi r}\,\ddot{p}(t_0)\sin\theta\;\hat{\theta}\end{aligned}}$$

$\hat{r}\times(\ddot{\vec{p}}\times\hat{r})$ is in $-\hat{\theta}$ direction

note: $\ddot{\vec{p}}(t_0)$ is necessarily a real valued vector, so we can always choose it to be along $\hat{z}$

$\ddot{p} = |\ddot{\vec{p}}|$

Similarly $\vec{B}(\vec{r},t) = \int d\omega\;\vec{B}(\vec{r},\omega)\,e^{-i\omega t}$

$$= -\frac{\mu_0}{4\pi c r}\int d\omega\; e^{-i\omega(t-r/c)}\,\omega^2\,\vec{p}(\omega)\times\hat{r}$$

$$= \frac{\mu_0}{4\pi c r}\frac{\partial^2}{\partial t^2}\left[\int d\omega\; e^{-i\omega(t-r/c)}\,\vec{p}(\omega)\times\hat{r}\right]$$

$$\boxed{\begin{aligned}\vec{B}(\vec{r},t) &= \frac{\mu_0}{4\pi c r}\,\ddot{\vec{p}}(t_0)\times\hat{r} \\ &= \frac{\mu_0}{4\pi c r}\,\ddot{p}(t_0)\sin\theta\;\hat{\phi}\end{aligned}}$$

with $\ddot{\vec{p}}(t_0)$ taken along the $\hat{z}$ axis

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}\left(\frac{\mu_0}{4\pi r}\ddot{p}(t_0)\sin\theta\right)^2 \underbrace{(\hat{\theta}\times\hat{\phi})}_{=\hat{r}}$$

$$\vec{S} = \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2 \frac{\sin^2\theta}{r^2}\,\hat{r}$$

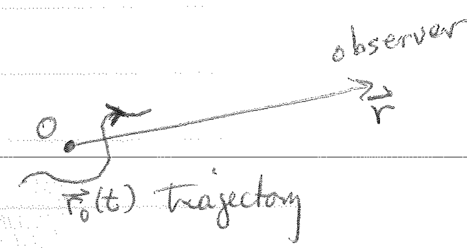
<u>total power</u> radiated through a sphere of radius r is

$$P = \oint d\vec{a}\cdot\vec{S} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, r^2 \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2 \frac{\sin^2\theta}{r^2}$$

$$= \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2 2\pi \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{\frac{4}{3}}$$

$$= \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2 \qquad \text{use } \mu_0 = \frac{1}{\varepsilon_0 c^2}$$

$$\boxed{P = \frac{1}{4\pi\varepsilon_0}\frac{2\left[\ddot{p}(t_0)\right]^2}{3c^3}}$$



For a <u>point charge moving</u> along a trajectory $\vec{r}_0(t)$

$$\vec{p}(t) = q\vec{r}_0(t) \Rightarrow \ddot{\vec{p}}(t) = q\ddot{\vec{r}}_0(t) = q\vec{a}(t)$$

acceleration

$$\boxed{P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}q^2\frac{a^2(t_0)}{c^3}}$$

← total power passing through sphere of radius r at time t is due to acceleration at retarded time t_0

<u>Larmor's formular</u>

power radiated ~ (acceleration)^2

moving point charge: <u>fields</u> + <u>Poynting vector</u>

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi c r} q\,\vec{a}(t_0)\times\hat{r}$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{c^3}\frac{\vec{a}(t_0)\times\hat{r}}{r}}$$ using $\mu_0 = \frac{1}{\varepsilon_0 c^2}$

$$\vec{E}(\vec{r},t) = -\frac{\mu_0 q}{4\pi r}\hat{r}\times(\vec{a}(t_0)\times\hat{r})$$

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{c^2}\frac{\hat{r}\times(\hat{r}\times\vec{a}(t_0))}{r}}$$

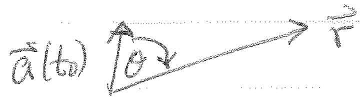
Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{-1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{c^5 r^2}\underbrace{\left[\hat{r}\times(\hat{r}\times\vec{a})\right]\times\left[\hat{r}\times\vec{a}\right]}_{\text{use triple product rule}}$$

$$= \frac{1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{c^5 r^2}\left\{\hat{r}\underbrace{(\hat{r}\times\vec{a})^2}_{a^2-(\hat{r}\cdot\vec{a})^2} - (\hat{r}\times\vec{a})\underbrace{\hat{r}\cdot(\hat{r}\times\vec{a})}_{=0}\right\}$$

$$\vec{S} = \frac{1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{c^5 r^2}\left(a^2-(\hat{r}\cdot\vec{a})^2\right)\hat{r}$$

$\vec{a}$ evaluated at t_0



$$\vec{S} = \frac{1}{\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2 a^2(t_0)}{c^5 r^2}\sin^2\theta\,\hat{r}$$

use $\mu_0\varepsilon_0 = \frac{1}{c^2}$

$$\boxed{\vec{S} = \frac{1}{4\pi\varepsilon_0}\frac{q^2 a^2(t_0)}{4\pi c^3 r^2}\sin^2\theta\,\hat{r}}$$

Radiation – Reaction Force – radiation damping
(Griffiths ~~§9.5~~ 11.2.2, Feynman v II Chpt 28, Jackson chpt 17)

So far, we considered problems of two types:
1) given ρ and $\vec{J}$, find $\vec{E}$ and $\vec{B}$
2) given $\vec{E}$ and $\vec{B}$, find forces on ρ and $\vec{J}$, and compute motion of charges

Now we combine the two ~~pro~~ types of problems.
~~Now~~ We've seen that charges produce $\vec{E}$ and $\vec{B}$ fields, and accelerating charges radiate $\vec{E}$ and $\vec{B}$ fields in the form of electromagnetic waves. These radiated fields can then act back on the charge, and effect the charges subsequent motion. This force on the charge, due to its own radiated $\vec{E}$ and $\vec{B}$ fields, is called the "radiation – ~~react~~ reaction force". Another way to see that there should be such a force: as a charge accelerates, it radiates away energy in the form of electromagnetic radiation – the resulting loss in energy should be reflected in a ~~change~~ decrease in the particles kinetic energy (via energy conservation) and hence in the particles velocity – hence the radiation of e-m waves should alter the charges ~~fut~~ future motion.

Thus problems (1) and (2) above, can never really be independent. The motion of charges is coupled back to the fields they produce, which in turn are determined by the charges motion! ~~Th~~ The self consistent solution to this coupled problem causes conceptual difficulties which ~~are~~ difficult to resolve. We will see however that these ~~pro~~ difficulties only arise on time and length scales where classical physics breaks down. Unfortunately, quantum mechanics has also not satisfactorily dealt with these difficulties.

To see when the radiation-reaction force will play an important role in the motion of a charged particle, consider following situations:

1) charge q, initially at rest, is acted on by force for a time T.
The change in kinetic energy after time T is

$$E_0 \sim m(aT)^2$$ where a is the average acceleration due to the force

From Larmor's formula, the <u>power</u> radiated away is $\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}$. So during the particles acceleration

it loses total energy $E_{rad} \sim \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}T$.

This radiation will play a negligible role in determining the particles motion (ie the radiation-reaction force can be considered as a <u>small</u> perturbation) when $E_{rad} \ll E_0$. Conversely, radiation plays a major role in the particles motion when $E_{rad} \gtrsim E_0$. This is true when

$$\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}T \gtrsim m(aT)^2$$

$$\text{or when} \quad T \lesssim \tau \equiv \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{mc^3}$$

for an electron $q = e$, $m = m_e$ (this gives largest possible τ)

$$\tau \simeq 6.26\times10^{-24}\ \text{sec}$$

(ie dramatically alter motion of charge)

So radiative effects are only important on time scales
$t < 6.26\times10^{-24}$ sec

8) Consider a charge q undergoing periodic motion with frequency ω_0, and amplitude of oscillation d.
Typical kinetic energy is

$$E_0 \simeq m\omega_0^2 d^2$$

$E_{rad} \gtrsim E_0$ when

$$\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}T \simeq \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}(\omega_0^2 d)^2\frac{1}{\omega_0} \gtrsim m\omega_0^2 d^2$$

(where we used $T \sim \frac{1}{\omega_0}$, $a \sim \omega_0^2 d$)

$$\Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\frac{\omega_0}{m} \gtrsim 1 \quad \text{or when } \omega_0 \gtrsim 1/\tau$$

where $\tau = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3 m} = 6.26\times10^{-24}$ sec for electron

radiation-reaction a non-perturbative effect only for frequencies $\omega_0 \gtrsim 1/\tau$ | as in (1) to see how small τ is, distance traveled by light in time τ is

$$c\tau = 2\times10^{-13}\,\text{cm} \simeq \text{typical nuclear radius of a heavy atom}$$

freq of oscillator where radiation-reaction is non-perturbative is $\omega_0 \sim 1/\tau = 0.16\times10^{+24}\,\text{sec}^{-1}$

$\gg$ freq of X-rays ($10^{18}\,\text{sec}^{-1}$)

$\gg$ freq of atomic spectra ($10^{15}\,\text{sec}^{-1}$)

From Heisenberg uncertainty principle, if we localized charge in time to within a certainty τ, we would have uncertainty in energy of

$$\Delta E \sim \hbar/\tau \sim \hbar\omega_0 = \frac{6.58\times10^{-16}\,\text{ev-sec}}{6.26\times10^{-24}\,\text{sec}}$$

$$= 10^{8}\,\text{ev} = 100\,\text{Mev} = 200\times \text{rest mass of electron}$$

$\simeq$ pion rest mass - nuclear field quanta

All the above implies that radiation-reaction effects remain <u>small</u> corrections to a charges motion, except when one gets down to <u>nuclear</u> time and length scales. Therefore in <u>classical</u> problems, it is usually a very good approximation to ignore self-force effects in computing the motion of charges. - radiation-reaction effects can be added as a small perturbation.

radiation - reaction effects are small even on atomic scales

$$\left(\begin{array}{l} \text{Bohr radius} \simeq .5\times10^{-8}\,\text{cm} \gg c\tau \simeq 10^{-13}\,\text{cm} \\ \text{atomic energy} \simeq 13.6\ \text{ev} \ll \Delta E = 200\,\text{Mev} \end{array} \right)$$

Therefore, consider classical model of (an atom, with the) electron orbiting the nucleus, as planets orbit the sun. Energy radiated in one period of revolution $\ll$ average electron kinetic energy $\Rightarrow$ loss of radiated energy causes electron to slowly spiral in towards the nucleus. Lets estimate how long it takes for electron in ground state of hydrogen, ~~to lose all its kinetic energy to radiation, and so~~ to spiral into nucleus.

For no radiation: centripetal acceleration is $\vec{a} = -\frac{v^2}{r}\hat{r}$

force is $\frac{-1}{4\pi\epsilon_0}\frac{e^2}{r^2}\hat{r}$

Newton's equ $\Rightarrow \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r}$

$\Rightarrow \frac{1}{2}mv^2 = \frac{e^2}{4\pi\epsilon_0\, 2r}$ kinetic energy in orbit of radius r

energy in orbit of radius r is

$$\underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic}} + \underbrace{\frac{-1}{4\pi\epsilon_0}\frac{e^2}{r}}_{\text{potential}} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2r}$$

freq of oscillation $\omega_0 = \frac{v}{r} = \sqrt{\frac{e^2}{4\pi\epsilon_0 m r^3}}$

For an electron $-e$ in classical orbit around nucleus of charge $+e$

$$\vec{F} = m\vec{a} \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r} \Rightarrow \frac{1}{2}mv^2 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r}$$

Total energy of charge in orbit at radius r is

$$E(r) = \underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic}} + \underbrace{V(r)}_{\text{potential}} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r}$$

Now consider the energy lost due to radiation. From Larmor

$$\frac{dE}{dt} = -P = -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c^3}a^2 \qquad \text{use } a = \frac{v^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2 m}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c^3}\left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\frac{1}{m}\right)^2$$

Also

$$\frac{dE}{dt} = \frac{dE}{dr}\frac{dr}{dt} = \frac{d}{dr}\left(\frac{-1}{4\pi\varepsilon_0}\frac{e^2}{2r}\right)\frac{dr}{dt} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r^2}\frac{dr}{dt}$$

$$\Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r^2}\frac{dr}{dt} = -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c^3}\left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\frac{1}{m}\right)^2$$

$$\frac{dr}{dt} = -\frac{4}{3}\frac{1}{c^3}\left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{m}\right)^2\frac{1}{r^2}$$

use $\tau \equiv \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{mc^3}$

$$\frac{dr}{dt} = \frac{-4}{3}\frac{1}{c^3}\left(\frac{3}{2}c^3\tau\right)^2\frac{1}{r^2} = -\frac{3c^3\tau^2}{r^2}$$

$$\boxed{\frac{dr}{dt} = -\frac{3c^3\tau^2}{r^2}}$$

determines how radius of orbit decreases in time as energy gets radiated away!

~~use~~ $\cancel{\frac{v^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mr^2}}$

$\cancel{\frac{dr}{dt} = -\frac{4}{3c^3}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{mr^2}\right)^2 r^2}$ ~~use~~ $\cancel{\tau = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2}{c^3 m}}$

$$\boxed{\frac{dr}{dt} = -\frac{3c^3\tau^2}{r^2}}$$ = equation of motion for radius of orbit

integrate to get solution

$$\int_{r_0}^{r(t)} dr\, r^2 = -\int_0^t dt\; 3c^3\tau^2$$

$$\frac{r^3(t) - r_0^3}{3} = -3c^3\tau^2 t$$

$$\boxed{r^3(t) = r_0^3 - 9c^3\tau^2 t}$$ r^3 decreases linearly in time

above result is only good as long as $r(t) \gtrsim c\tau$.
But $c\tau$ is the nuclear size. So we can use above equation for decay of electrons orbit, all the way until the electron crashes into the nucleus!

electron crashes into nucleus when

$$r^3(t) \approx 0 = r_0^{\,3} - 9c^3\tau^2 t$$

this happens in a time

$$t = \frac{r_0^{\,3}}{9c^3\tau^2} = \cancel{\frac{r_0^3 \tau}{9c^3\tau^3}} = \left(\frac{r_0}{c\tau}\right)^3 \frac{\tau}{9}$$

use Bohr radius for $r_0 = .5 \times 10^{-8}$ cm

$$c\tau = 2 \times 10^{-13} \text{ cm}$$

$$\tau = 6 \times 10^{-24} \text{ sec}$$

$$t = \left(\frac{.5 \times 10^{-8}}{2 \times 10^{-13}}\right)^3 \frac{6 \times 10^{-24}}{9} = \left(\frac{1}{4}\right)^3 \times 10^{15} \times \frac{6 \times 10^{-24}}{9}$$

$$\boxed{t \simeq 10^{-11} \text{ sec}}$$

So if classical mechanics + E+M held on atomic scale, the lifetime of the hydrogen atom would be 10^{-11} sec.

The experimentally observed stability of the hydrogen atom, as well as the exptily observed <u>absence</u> of radiation from hydrogen ~~at~~ in its ground state, were some of the motivating facts that lead to the discovery of <u>quantum mechanics</u>!