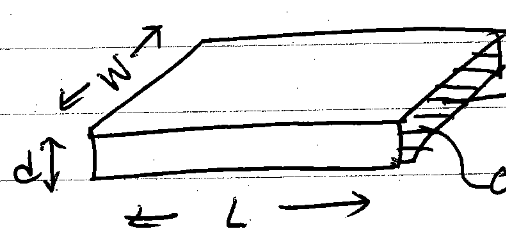
Why two dimensions is important

In a theoretical model, it is resistivity ρ_{xx} and ρ_{xy} that one computes. These are the quantities that are intrinsic to the material and the physical situation. The total Resistance R_{xx}, R_{xy} are related to ρ_{xx} and ρ_{xy} by sample geometry. However, while theory calculates ρ, it is R that is directly measured in experiment. So to convert from measured R to prediction for ρ it would seem that one needs to know very precisely the correct geometrical factors for this conversion.
Not true in two dimensions. This is important because the theory predicts that ρ_{xy} is quantized in units of h/se^2, s integer, whereas it is R_{xy} that is the directly measured quantity. We want to be able to test the theory without worrying about the geometry! To illustrate this, consider a simple rectangular geometry

in 3D



for geometry in which current flows along $\hat{x}$ and so $j_y = 0$, we have

$$\begin{pmatrix}E_x\\E_y\end{pmatrix} = \begin{pmatrix}\rho_{xx} & \rho_{xy}\\ -\rho_{xy} & \rho_{xx}\end{pmatrix}\begin{pmatrix}j_x\\0\end{pmatrix} = \begin{pmatrix}\rho_{xx} j_x\\ -\rho_{xy} j_x\end{pmatrix}$$

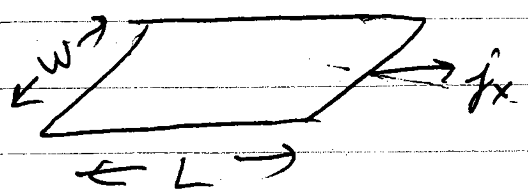
$$\rho_{xx} = \frac{E_x}{j_x} \qquad \rho_{xy} = \frac{-E_y}{j_x}$$

So then $R_{xx} = \dfrac{V_x}{I_x} = \dfrac{E_x L}{j_x A} = \rho_{xx}\left(\dfrac{L}{A}\right)$

$$R_{xy} = \frac{-V_y}{I_x} = \frac{-E_y W}{j_x A} = \rho_{xy}\left(\frac{W}{A}\right)$$

We see that to convert the measured resistance R_{xx}, R_{xy} to resistivities ρ_{xx}, ρ_{xy} we need the geometrical factors (L/A) and (W/A)

In 2D



Now j_x is a "sheet current" i.e. current per unit width
total current $I_x = j_x W$

So we have
$$R_{xx} = \frac{V_x}{I_x} = \frac{E_x L}{j_x W} = \rho_{xx}\left(\frac{L}{W}\right)$$

but

$$R_{xy} = \frac{-V_y}{I_x} = \frac{-WE_y}{Wj^x} = \frac{-E_y}{j^x} = \rho_{xy}$$

So $\boxed{R_{xy} = \rho_{xy}}$ in 2D

R and ρ have the same units in 2D and there is no geometrical factor relating R_{xy} to ρ_{xy}! Direct measurement of R_{xy} gives the quantized ρ_{xy}.

Note the units of $\rho_{xy} = h/se^2$ in 2D have dimensions of $\frac{h}{e^2} = \frac{\text{energy} \cdot \text{sec}}{\text{energy} \cdot \text{length}} = \frac{\text{sec}}{\text{length}}$

while the units of R in 2D have dimensions of

$$R = \frac{V}{I} = \frac{\text{energy}/\text{charge}}{\text{charge}/\text{sec}} = \frac{\text{energy} \cdot \text{sec}}{(\text{charge})^2}$$

$$= \frac{\text{energy} \cdot \text{sec}}{\text{energy} \cdot \text{length}} = \frac{\text{sec}}{\text{length}}$$

so $\frac{h}{e^2}$ has units of total resistance = units of resistivity only in two dimensions.

The conductance tensor $\overleftrightarrow{G} = \overleftrightarrow{R}^{-1}$ inverse of resistance tensor. In quantum Hall state with $\rho_{xx} = \sigma_{xx} = 0$,

$$-G_{xy} = \frac{1}{R_{xy}} = \frac{1}{\rho_{xy}} = -\sigma_{xy}$$

$G_{xy} = \sigma_{xy}$ Conductance and conductivity have the same units, and transverse conductance = transverse conductivity

<u>But</u> .... why in this case should ρ_{xy} have its classical Drude value

$$\frac{H}{nec} = \frac{n\phi_0}{\nu nec} = \frac{h}{\nu e^2} \ ?$$

Why can't quantum effects lead to something different for ρ_{xy}?

For the case of a ~~...~~ completely filled Landau level, one can directly compute quantum mechanically the current j_x and j_y. For $\vec{H} = H\hat{z}$ and total $\vec{E} = E\hat{y}$

$$j_y = -\frac{e}{m}\sum_{n,s}\sum_{k_x}\left\langle \psi_{nk_x}\right|\frac{\hbar}{i}\frac{\partial}{\partial y}\left|\psi_{nk_x}\right\rangle$$

$$j_x = -\frac{e}{m}\sum_{n,s}\sum_{k_x}\left\langle \psi_{nk_x}\right|\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{e}{c}Hy\left|\psi_{nk_x}\right\rangle$$

where $\psi_{nk_x} = e^{ik_x x}\,\phi_n(y-y_0)$ with

$y_0 = \frac{1}{\omega_c}\left[\frac{\hbar k_x}{m} - \frac{cE}{H}\right]$ is the ~~eigen~~ wave function

for the eigenstate in Landau level n, with eigenvalue k_x (look back at Landau level notes)

velocity operator is $\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}$

So

$$m v_x^{op} = \frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{e}{c}Hy \qquad \text{using } \vec{A} = -\hat{x}Hy$$

$$m v_y^{op} = \frac{\hbar}{i}\frac{\partial}{\partial y}$$

The sum on k_x is over all allowed values of k_x from 0 to $k_{x\,max} = \frac{L_y m \omega_c}{\hbar}$, and the sum on n, s is over all filled Landau levels (n integer, $s = \pm 1$ for spin up and down)

ϕ_n is the n^{th} harmonic oscillator eigenstate

You will do this for homework!

You will find $j_y = 0$ and $j_x = -\frac{necE}{H}$

Since $\vec{E} = E\hat{y}$, this gives

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} \begin{pmatrix} 0 \\ E \end{pmatrix} = \begin{pmatrix} \sigma_{xy} E \\ \sigma_{xx} E \end{pmatrix}$$

$$j_y = 0 \Rightarrow \sigma_{xx} = \boxed{\rho_{xx} = 0}$$

$$j_x = -\frac{necE}{H} \Rightarrow \sigma_{xy} = -\frac{1}{\rho_{xy}} = -\frac{nec}{H}$$

$$\Rightarrow \boxed{\rho_{xy} = \frac{H}{nec}} = \frac{n\phi_0}{\nu nec} = \frac{h}{\nu e^2}$$

So this calculation confirms the observed values of ρ_{xx} and ρ_{xy} at $H = \frac{n\phi_0}{\nu}$.

Note: We can only do the above calculation of ~~for~~ j_x and j_y for the special case of completely filled Landau levels ie no partially filled Landau levels.

In this case, since there is no electron scattering, the occupation function is just its equilibrium value, which at low $k_B T \ll \hbar\omega_c$ is just 1 for a state in the filled Landau level, and 0 for a state in an empty Landau level.

If we had a partially filled Landau level, the applied fields would cause electrons to scatter (scattering now is allowed in a partially filled Landau level since there are degenerate unoccupied states) and this would create some non-equilibrium steady state probability distribution for describing the probability that a given state in the Landau level is occupied or not. We would then need some theory of how to compute this non-equilibrium distribution in order compute the average j_x and j_y.

But we still have not answered question ②. Why are there plateaus in ρ_{xy}?

Why are there the plateaus?

We might expect that for $H \neq n\phi_0/s$ there is a partially filled Landau level $\Rightarrow$ there are empty states right at the Fermi energy $\Rightarrow$ scattering can take place $\rightarrow \rho_{xx} \neq 0$ and no reason for ρ_{xy} to remain fixed at h/se^2.

It ~~[illegible]~~ turns out that to explain the plateaus we need to assume that the electron gas sees a random potential due to static impurities in the material. It is perhaps paradoxical that to observe best the quantization of conductance e^2/h we need a dirty system and not a pure one, but this turns out to be true!

To explain the effect we need to discuss the phenomenon of "localization" of electron wavefunctions in the presence of a random potential. Perturbation theory would suggest that when a random potential is added to the system, the degenerate Landau level states will split into an energy band of finite width (In general, perturbations that obey no particular symmetry will split degenerate energy levels). But in this case the effect of the random potential is even more dramatic. Not only does it split the degenerate states of a given Landau level into a band of finite width in energy, but the states in the low and high energy tails of this band become localized; states in the middle of the band remain extended.

Our old eigenstates of the pure problem (no random potential) were

$$\psi_{nk} \sim e^{ikx} \phi_n(y - y_0(k))$$

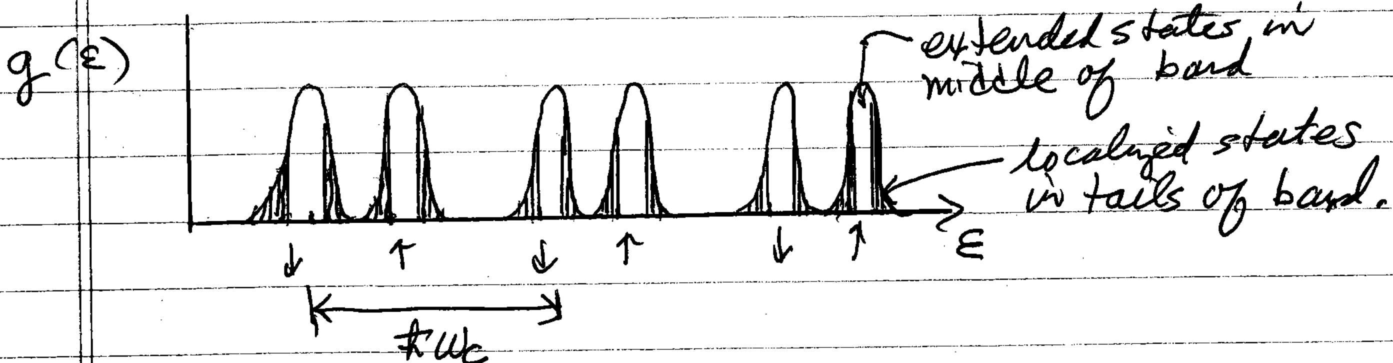
Such states are extended because there is a finite probability $|\psi|^2$ to find the electron at any position x in the system. The e^{ikx} part travels through the entire system

Impurities, however, create local potential wells that can trap electrons in bound states

$$\psi(\vec{r}) \sim e^{-|\vec{r}-\vec{r}_0|/l}$$

in which the electron is localized to position $\vec{r}_0$ with a localization length scale l.

In the presence of a random potential, the density of states of the 2D electron gas in a uniform magnetic field $H\hat{z}$ will look like



Now, as H increases or decreases from $n\Phi_0/A$, the Fermi energy ε_F (which lies in the gap between the Landau level bands at $H = n\Phi_0/A$) just enters the region of <u>localized</u> states. Localized states <u>cannot</u> carry any current because the electron is bound to a particular impurity site and not free to travel throughout the sample (at least this is true at low temperature) $\Rightarrow$ j_x and j_y do not change from the values they had at $H = n\Phi_0/A$.

~~$\rho_{xy} = E_y / j_x$ does not change~~

~~since E_y is fixed experimentally and j_x does not change~~

~~plateaus in ρ_{xy}~~

If $\vec{E}$ is held fixed experimentally, and $\vec{j}$ does not change as $\vec{H}$ is varied, then by

$$\vec{j} = \overleftrightarrow{\rho} \cdot \vec{E} \quad \Rightarrow \overleftrightarrow{\rho} \text{ does not change}$$

$$\Rightarrow \text{plateaus in } \rho_{xy} \text{ and } \rho_{xx}$$

__Problem:__ Above argument shows why there are plateaus in ρ_{xx} and ρ_{xy}. But now that we have added impurities and the degenerate Landau level states have changed into a band of finite energy width with both extended and localized states, how do we know that we still have $\rho_{xx} = 0$ and $\rho_{xy} = \frac{h}{s e^2}$? That calculation was done for the __pure__ system!

__Answer:__ gauge invariance argument due to Laughlin shows that $\rho_{xx} = 0$ and $\rho_{xy} = h/se^2$ even in the random case.