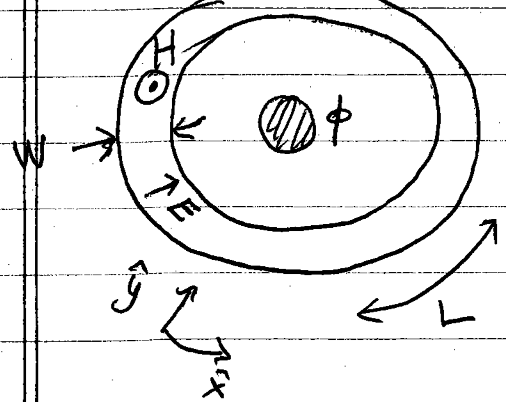
Consider the following geometry



strip is now a ring

$\hat{x}$ is the circumferential direction

$\hat{y}$ is the radial direction

L is circumference

W is width

magnetic field $H\hat{z}$ penetrates the strip

electric field $E\hat{y}$ along width of strip

Now add an infinite solenoid in the center with a magnetic flux ϕ through it. The magnetic field from the solenoid does <u>not</u> penetrate the strip. ~~But it does produce a~~ It is confined to the ~~center~~ inside of the solenoid. But it does produce a vector potential inside the strip that will effect the electron states there. Let $\Delta\vec{A}$ be this vector potential.

$$\Delta\vec{A} = \frac{\phi}{L}\hat{x} \qquad \text{follows from}$$

$$\oint d\vec{l}\cdot\vec{A} = \Delta A\, L$$

$$\parallel$$

$$\int d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = \int d\vec{a}\cdot\vec{H} = \phi$$

Now the Hamiltonian for the electrons i in the strip is

$$\mathcal{H} = \sum_i \left\{ \frac{1}{2m}\Big(\vec{p}_i + \underbrace{\frac{e}{c}\vec{A}}_{\substack{\text{vector potential}\\ \text{from } \vec{H}}} + \underbrace{\frac{e}{c}\Delta\vec{A}}_{\substack{\text{vector potential}\\ \text{from solenoid } \phi}}\Big)^2 + \underbrace{eEy_i}_{\substack{-eV(y_i)\\ \text{electrostatic}\\ \text{potential from } E\hat{y}}} \right\}$$

For $\Delta\vec{A} = \Delta A\hat{x}$ we have

$$\frac{\partial \mathcal{H}}{\partial \Delta A} = \frac{e}{cm}\sum_i \left(p_{ix} + \frac{e}{c}A_x + \frac{e}{c}\Delta A\right) = \frac{e}{c}\sum_i v_{ix}$$

v_{ix} is x-component of velocity of electron i

Let $U = \langle \mathcal{H} \rangle$ average over occupied electron states

Then $\dfrac{\partial U}{\partial \Delta A} = \dfrac{e}{c} N \langle v_x \rangle$ $\quad$ N = total number electrons
$\langle v_x \rangle$ = average electron velocity

The total current flowing in the $\hat{x}$ direction is therefore

$$I_x = -en\langle v_x\rangle W = -\frac{eN}{LW}\langle v_x\rangle W = -\frac{eN}{L}\langle v_x\rangle$$

$$\Rightarrow \frac{\partial U}{\partial \Delta A} = \frac{eN}{c}\langle v_x\rangle = -\frac{L}{c} I_x$$

$$\Rightarrow -\frac{c}{L}\frac{\partial U}{\partial \Delta A} = \boxed{-c\frac{\partial U}{\partial \phi} = I_x}$$

where we used $L\Delta A = \phi$

If the average energy U depends on ϕ, then a current I_x will flow.

How does adding the vector potential $\Delta\vec{A}$ effect the eigenstates of $\mathcal{H}$?

Since $\Delta\vec{A}$ is a constant, adding $\Delta\vec{A}$ can be viewed as making a gauge transformation of $\vec{A}$ (where $\vec{A}$ gives the $\vec{H}$ penetrating the strip)

$$A' = \vec{A} + \vec{\nabla}\Lambda \quad \text{where } \Lambda = \Delta A\, x$$

under this gauge transform wavefunctions transform as

$$\psi' = \psi e^{-ie\Lambda/\hbar c} = \psi e^{-ie\Delta A\, x/\hbar c}$$

If ψ is a localized state, $\psi \propto e^{-|\vec{r}-\vec{r}_0|/\ell}$, then ψ' has the same energy as ψ, and contributes nothing to the current, as before.

If ψ is an extended state however, adding $\Delta\vec{A}$ causes a problem with our requirement of periodic boundary conditions: we have $\psi(x+L) = \psi(x)$ so

$$\begin{aligned}\psi'(x+L) &= \psi(x+L)\, e^{-ie\Delta A(x+L)/\hbar c}\\ &= \psi(x)\, e^{-ie\Delta A\, x/\hbar c}\, e^{-ie\Delta A L/\hbar c}\\ &= \psi'(x)\, e^{-ie\Delta A L/\hbar c}\end{aligned}$$

Unless $\frac{e\Delta A L}{\hbar c} = 2\pi n$, n integer, ψ' will no longer satisfy periodic boundary conditions.

$$\frac{e\Delta A L}{\hbar c} = 2\pi n \implies \Delta A L = \phi = 2\pi n \frac{\hbar c}{e} = n\left(\frac{hc}{e}\right)$$
$$= n\phi_0$$

So only if $\phi = n\phi_0$ will the eigenstates we found for $\phi = 0$ continue to be good eigenstates obeying periodic boundary conditions when $\phi > 0$. (This is not a problem for the localized states since they do not extend across the entire system and so are unaffected by boundary conditions)

To see what is going on, lets consider the pure model where the eigenstates of the Landau levels are

$$\psi_{nk} = e^{ikx}\phi_n(y - y_0(k)) \qquad y_0(k) = \frac{c}{eH}\left(\hbar k - \frac{eEm}{H}\right)$$

ϕ_n is n^{th} eigenstate of the harmonic oscilator

For periodic boundary conditions along $\hat{x}$ we required $k = \frac{2\pi}{L}\,l$ with l integer

Now after turning on ϕ in solenoid, gauge invariance gives as the new eigenstate

$$\psi'_{nk} = \psi_{nk}\, e^{-ie\Delta A\, x/\hbar c} = e^{i\left(k - \frac{e\Delta A}{\hbar c}\right)x}\phi_n(y - y_0(k))$$

If we want ψ'_{nk} to obey periodic boundary conditions it is now necessary that k satisfies

$$k - \frac{e\Delta A}{\hbar c} = \frac{2\pi}{L} l \quad , \quad l \text{ integer}$$

$$\Rightarrow k = \frac{2\pi l}{L} + \frac{e\Delta A}{\hbar c}$$

Now the allowed values of $y_0(k)$ are

$$y_0 = \frac{c}{eH}\left(\hbar k - \frac{eEm}{H}\right) = \frac{c}{eH}\left(\frac{2\pi\hbar}{L} l - \frac{eEm}{H}\right) \text{ for } \phi = 0$$

but

$$y_0 = \frac{c}{eH}\left(\frac{2\pi\hbar}{L} l + \frac{e\Delta A}{c} - \frac{eEm}{H}\right) \quad \text{for } \phi \geq 0$$

we therefore see that there is a shift in the allowed values of y_0

$$\Delta y_0 \equiv y_0(\phi) - y_0(\phi=0) = \left(\frac{c}{eH}\right)\left(\frac{e\Delta A}{c}\right) = \frac{\Delta A}{H}$$

So the new eigenstates in the presence of ϕ are the same as the old ones, except with a k shifted by $\frac{e\Delta A}{\hbar c}$ and a corresponding y_0 shifted by $\frac{\Delta A}{H}$

Recall y_0 gives the position along $\hat{y}$ at which the harmonic oscillator part of the wavefunction is centered, ie $\phi_n(y-y_0)$.

Recall now that in a finite electric field $E_y\hat{y}$, the energy of the eigenstate ψ_{nk} is

$$\varepsilon_{nk} = \hbar\omega_c(n+1/2) + eE_y y_0(k) + \frac{e^2E_y}{2m\omega_c}$$

so if $y_0(k)$ shifts by $\Delta y_0 = \frac{\Delta A}{H}$ when ϕ turned on, then ε_{nk} shifts by $\Delta\varepsilon = eE_y\Delta y_0 = eE_y\frac{\Delta A}{H}$.
This is what gives the dependence of $U = \langle\mathcal{H}\rangle$ on ϕ that leads to the current I_x.

For s completely filled Landau levels, with $N = s(LW)\left(\frac{H}{\Phi_0}\right)$ total electrons, the change in U is

$$\Delta U = N\cdot\frac{eE_y\Delta A}{H} = s\,LW\frac{H}{\Phi_0}\frac{eE_y\Delta A}{H}$$

$$= s\,eW\frac{E_y\phi}{\Phi_0} \quad \text{using } L\Delta A = \phi$$

so $$I_x = -c\frac{\partial U}{\partial\phi} = -\frac{s\,ceWE_y}{\Phi_0} = -\frac{s\,ceWE_ye}{hc}$$

$$= -s\frac{e^2}{h}WE_y$$

Now $I_x = j_xW$ so $j_x = -s\dfrac{e^2}{h}E_y$

and so $$\boxed{\rho_{xy} = -\frac{E_y}{j_x} = \frac{h}{se^2}}$$

the quantized values (same as classical Drude result)

Now look at what happens when ... increased to equal Φ_0. Then the shift in y_0 is

$$\Delta y_0 = \frac{\Delta A}{H} = \frac{(\Phi_0/L)}{H} = \frac{\Phi_0}{LH} \qquad \text{since } \Delta A = \frac{\Phi_0}{L}$$

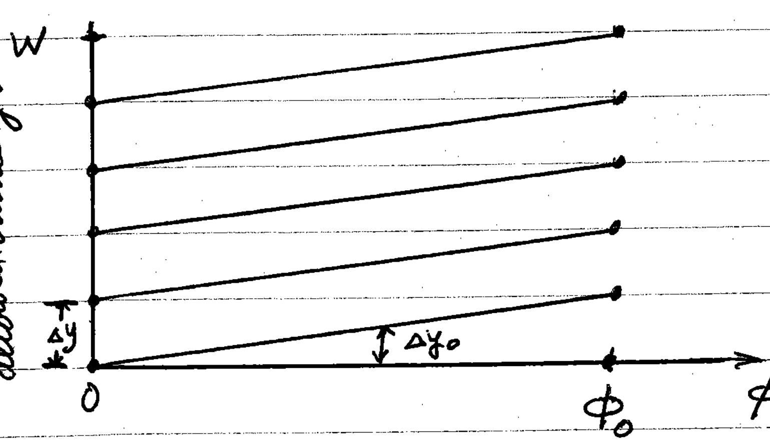
compare this to the spacing in allowed values of y_0 due to the spacing between allowed values of k

$$y_0 = \frac{c}{eH}\left(\hbar k - \frac{eEm}{H}\right)$$

$$\Delta k = \frac{2\pi}{L} \quad \Rightarrow \quad \Delta y = \frac{c}{eH}\hbar \Delta k = \frac{c\hbar}{eH}\frac{2\pi}{L}$$

$$= \frac{hc}{eHL} = \frac{\Phi_0}{HL}$$

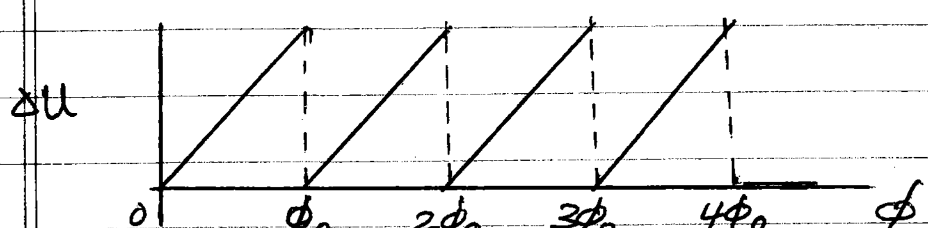
So when $\phi \to \Phi_0$, the shift in y_0 due to ϕ, Δy_0, becomes equal to the spacing between allowed values of y_0, Δy. Pictorially the situation is as sketched below:



As $\phi \to \Phi_0$, the set of allowed $\{y_0\}$ maps back onto ~~themsel~~ itself with each y_0 moving into the one above it. The $y_0 = W$ at $\phi = 0$ becomes $y_0 = 0$ at $\phi = \Phi_0$

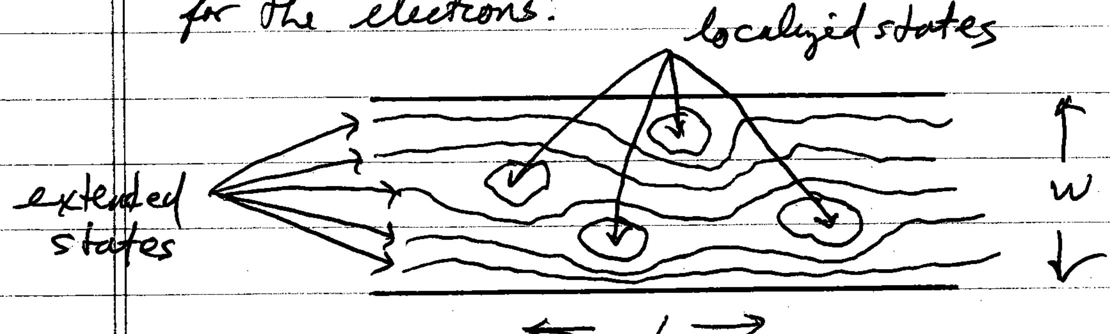
For a filled Landau level, the process of ϕ going from zero to Φ_0 results in a net motion of one electron transferred across the width of the strip.

As $\phi=0$ increases to $\phi=\phi_0$, all the eigenstates map back onto themselves. This is consistent with what we found by the gauge ~~invariance~~ transformation: when $\phi=\phi_0$, then $\psi' = \psi e^{i2\pi}$. The energy as a function ϕ thus looks like:



$\Delta U = U(\phi) - U(0)$

Above was for the pure (impurity-free) case. What happens when we add a random potential for the electrons?



When $\phi=0$ increases to $\phi=\phi_0$, the eigenstates of the system must map back onto themselves. The gauge transformation argument shows this must be true even when there is a random potential. As ϕ increases, the localized states do not change, but the extended states must change by increasing their effective value of "y_0". As $\phi=0$ increases to $\phi=\phi_0$, the extended state at some average y_0 maps onto the one above it at $y_0+\Delta y$. For a filled Landau level as $\phi = 0$ increases to $\phi = \phi_0$ the net result is that an integer number of electrons have been transported across the width W of the system. Each such electron increases the energy by eE_yW, so the total change in energy for a set of filled Landau levels is

$$\Delta U = s e E_y W \qquad s \text{ integer}$$

So as in the pure case

$$I_x = -c \frac{\Delta U}{\Delta \phi} = -c \frac{s e E_y W}{\Phi_0} = -\frac{c s e E_y W}{hc/e}$$

$$\frac{I_x}{W} = j_x = -s \frac{e^2}{h} E_y$$

$$\text{so} \quad \boxed{\rho_{xy} = \frac{-E_y}{j_x} = \frac{h}{se^2}}$$

So we have now shown that the random potential leads to plateaus in ρ_{xy} and ρ_{xx} that are fixed at the values they have at $H = n\Phi_0/s$ (corresponding to s filled Landau levels) and the the value of ρ_{xy} here is $\rho_{xy} = h/se^2$. Since the Landau levels are completely filled, there is no scattering, so also $\rho_{xx} = 0$ as in the pure case.

The key to Laughlin's argument is that as $\phi = 0$ increases to $\phi = \Phi_0$, gauge invariance requires that the eigenstates must map back onto themselves. This can only happen by transporting an integer number of electrons across the width of the strip W, thus quantizing ΔU.