Magnetostatics

Lorentz Force

a charge q, in motion with velocity $\vec{v}$, feels the force

$$\vec{F} = q(\vec{E} + k_4 \, \vec{v} \times \vec{B}) \quad \leftarrow \text{Lorentz force}$$

$\vec{B}$ is the <u>magnetic field</u> at the position of the charge.
k_4 is a universal constant.

Just as the constant k_1 fixed the units of charge q, the constant k_4 can be viewed as fixing the units of B magnetic field. By choosing the units of q and B appropriately, we are free to choose <u>any</u> values for k_1 and k_4.

Magnetic field $\vec{B}$ is generated by <u>moving charge</u>.
A charge q' with velocity $\vec{v}'$ ($v' \ll c$) located at the origin $\vec{r}' = 0$ produces a magnetic field at position $\vec{r}$.

holds only nonrelativistically $\longrightarrow$ $$\vec{B}(\vec{r}) = k_5 \, q' \frac{\vec{v}' \times \vec{r}}{r^3} = \frac{k_5}{k_1} \, \vec{v}' \times \vec{E}(\vec{r})$$

k_5 is a universal constant. We will see that it <u>cannot</u> be chosen independently of k_1 and k_4. (since k_1 fixed units of q, and k_4 fixed units of $\vec{B}$, there are no further new quantities whose units could be adjusted to allow us to fix k_5 arbitrarily)

The force on a charge q at position $\vec{r}$, moving with velocity $\vec{v}$, due to a charge q' at the origin moving with velocity $\vec{v}'$ is, in non-relativistic limit $(v, v' \ll c)$,

$$\vec{F} = k_1 q q' \frac{\vec{r}}{r^3} + k_4 k_5 q q' \vec{v} \times \frac{(\vec{v}' \times \vec{r})}{r^3}$$

Coulomb force — magnetic analog of Coulomb force

The magnetic part is just the point charge equivalent of the Biot-Savart law for the force between current ~~carry~~ carrying wires. If we regard $q\vec{v} = \vec{I}$ as the current of charge q, and $q'\vec{v}' = \vec{I}'$ as the current of charge q', then the magnetic force is $k_4 k_5 \, \vec{I} \times (\vec{I}' \times \frac{\vec{r}}{r^3})$ which is the Biot-Savart Law.

Rewrite above force as

$$\vec{F} = k_1 \left(1 + \frac{k_4 k_5}{k_1} \vec{v} \times \vec{v}' \times \right) \frac{\vec{r}}{r^3} q q'$$

we see that $\left(\frac{k_4 k_5}{k_1}\right)$ has units of $(\text{velocity})^{-2}$

it must be independent of whatever convention one used to choose the units of q or B (ie independent of choices for k_1 and k_4). Experimentally it is found that

$$\left(\frac{k_4 k_5}{k_1}\right) = \frac{1}{c^2} \qquad c = \text{speed of light in vacuum}$$

Continuum current density

For charges q_i at positions $\vec{r}_i(t)$ with $\vec{v}_i = \frac{d\vec{r}_i}{dt}$
we define the current density

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i(t)\, \delta(\vec{r}-\vec{r}_i(t))$$

units of $\vec{j}$ are $(\text{charge})\left(\frac{\text{length}}{\text{time}}\right)\left(\frac{1}{\text{length}^3}\right) = \left(\frac{\text{charge}}{\text{area}\cdot\text{time}}\right)$

charge per unit area per unit time

For a surface S

$$\int_S da\, \hat{n}\cdot\vec{j} = I$$ current (charge per unit time) passing through surface S

Charge Conservation vol V bounded by surface S

$$\frac{d}{dt}\int_V d^3r\, \rho(\vec{r},t) = -\oint_S da\, \hat{n}\cdot\vec{j}$$

rate of change of total charge in V = $(-)$ charge flowing out of V through S per unit time

use $\oint_S da\,\hat{n}\cdot\vec{j} = \int_V d^3r\, \vec{\nabla}\cdot\vec{j} = -\int_V d^3r\, \frac{\partial}{\partial t}\rho(\vec{r},t)$

$\Rightarrow$ local charge conservation

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

A static situation has $\frac{\partial \rho}{\partial t} = 0$

$\Rightarrow$ magnetostatics is defined by the condition $\underline{\vec{\nabla}\cdot\vec{J} = 0}$

Differential formulation of Biot-Savart

For a set of charges q_i at $\vec{r}_i$ we have

$$\vec{B}(\vec{r}) = \sum_i k_5\, q_i\, \vec{v}_i \times \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

$$= k_5 \int d^3r'\, \vec{J}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$= k_5 \int d^3r'\, \vec{J}(\vec{r}\,') \times \vec{\nabla}\left(\frac{-1}{|\vec{r}-\vec{r}\,'|}\right)$$

$$\vec{B}(\vec{r}) = k_5\, \vec{\nabla} \times \left[\int d^3r'\, \frac{\vec{J}(r')}{|\vec{r}-\vec{r}\,'|}\right]$$

where we used $\vec{\nabla}\times(\vec{A}\varphi) = -\vec{A}\times\vec{\nabla}\varphi$ when $\vec{A}$ is indep of $\vec{r}$

$\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ for any vector function $\vec{A}$

integral form $\oint_S da\, \hat{n}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{B} = k_5\, \vec{\nabla}\times\left[\vec{\nabla}\times\left(\int d^3r'\, \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

use $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$\vec{\nabla}\times\vec{B} = k_5\,\vec{\nabla}\left[\int d^3r'\;\vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\right]$$
$$-k_5\int d^3r'\;\vec{J}(\vec{r}')\,\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

in the 2nd term, $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$

in the 1st term, $\vec{\nabla}\cdot\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \vec{J}(\vec{r}')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

$$= -\vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

since $\vec{\nabla} = -\vec{\nabla}'$

So $\int d^3r'\,\vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) = -\int d^3r'\,\vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

integrate by parts
surface term $\to 0$ as
we take surface $\to\infty$
since $\vec{J}\to 0$ as $r\to\infty$

$$= \int d^3r'\left(\vec{\nabla}'\cdot\vec{J}(r')\right)\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

But for magnetostatics $\vec{\nabla}\cdot\vec{J} = 0 \Rightarrow$ only 2nd term remains

Thus, for magnetostatics

$$\vec{\nabla}\times\vec{B} = 4\pi k_5\,\vec{J} \qquad \text{Ampere's law}$$

integral form $\displaystyle\oint_C d\vec{l}\cdot\vec{B} = 4\pi k_5\int_S da\,\hat{n}\cdot\vec{J}$

curve bounding surface

Although above diff eq's were derived starting from a "non-relativistic" point-charge Biot-Savart law, the actually remain true for all magnetostatic situations

<u>So far</u> <u>electrostatics</u> $\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi k_1 \rho & \text{Gauss} \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$

<u>magnetostatics</u> $\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} & \text{Ampere} \end{cases}$

<u>current conservation</u> $\dfrac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

<u>Time Dependent situations</u>

<u>Faraday's law of induction</u> $\vec{\nabla}\times\vec{E} \neq 0$!

$$\oint_C d\vec{l}\cdot\vec{E} = -k_3 \frac{\partial}{\partial t}\int_S da\, \hat{n}\cdot\vec{B} \qquad \mathcal{E}_{mf} = -\frac{d\Phi}{dt}$$

emf around loop

mag flux

voltage around closed loop ~ −time rate of change of magnetic flux through loop

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -k_3 \frac{\partial\vec{B}}{\partial t}}$$

k_3 is universal constant

<u>Maxwell correction to Ampere's law</u>

In our derivation of $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J}$ we used $\vec{\nabla}\cdot\vec{J} = 0$. This is only true for magnetostatics – it is <u>NOT</u> true in general

Alternatively, since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ always, if Amperes law was true, we would necessarily conclude that $\vec{\nabla}\cdot\vec{J} = 0$. But $\vec{\nabla}\cdot\vec{J} = -\partial\rho \neq 0$ in general

Proposed correction: $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \vec{W}$

where $\vec{W}$ must be such that charge conservation holds.

Now

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = 4\pi k_5 \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{W}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{W} = -4\pi k_5 \vec{\nabla}\cdot\vec{J} = 4\pi k_5 \frac{\partial \rho}{\partial t} \quad \text{by charge conserv}$$

$$= \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} \quad \text{by Gauss law}$$

$$\Rightarrow \vec{W} = \frac{k_5}{k_1}\frac{\partial \vec{E}}{\partial t}$$

So corrected Amperes law is

$$\boxed{\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \frac{k_5}{k_1}\frac{\partial \vec{E}}{\partial t}}$$

⇒ EM waves

Now consider $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$

$= -\nabla^2\vec{B}$ as $\vec{\nabla}\cdot\vec{B} = 0$

If there are no sources ($\rho = 0$, $\vec{J} = 0$) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} = \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}$$

$$= -\frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2} \quad \text{by Faraday}$$

$$\nabla^2 B = \frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2} \quad \text{this is the wave equation}$$

$\Rightarrow \frac{k_5 k_3}{k_1}$ has units of (velocity)$^{-2}$

Since we know that the above wave equation describes electromagnetic waves, ie light, then

$$\frac{k_5 k_3}{k_1} = \frac{1}{c^2}$$

we already had $\quad \dfrac{k_4 k_5}{k_1} = \dfrac{1}{c^2}$

$$\Rightarrow k_3 = k_4$$

$\Rightarrow$ k_1 and k_4 are arbitrary - they can be chosen to be anything by adjusting the units of q and B. k_3 and k_5 are then fixed by $\dfrac{k_4 k_5}{k_1} = \dfrac{1}{c^2}$, $k_3 = k_4$

<u>Popular systems of E + M units</u>

	k_1	$k_3 = k_4$	k_5	
MKS or SI	$\frac{1}{4\pi\epsilon_0}$	1	$\frac{\mu_0}{4\pi}$	$\left(\epsilon_0\mu_0 = \frac{1}{c^2}\right)$
Gaussian or CGS	1	$\frac{1}{c}$	$\frac{1}{c}$	
Rationalized Gaussian	$\frac{1}{4\pi}$	$\frac{1}{c}$	$\frac{1}{4\pi c}$	

In MKS, charges are measured in "Coulombs"
current measured in "amps"
magnetic field measured in "tesla" = "weber/m^2"

In CGS, charges are measured in "statcoulombs"
current measured in "statamperes"
magnetic field measured in "gauss"

$1 \text{ tesla} = 10^4 \text{ gauss}$

We will use the CGS or Gaussian units

1) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ 3) $\vec{\nabla}\cdot\vec{B} = 0$

2) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

Maxwell's Equ's !!

$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$

Lorentz force

Physical content of Maxwell's equs.

1) Gauss Law for electric field – charge is source of $\vec{E}$ field. Field lines can begin and end at point charges

2) Faraday's Law of induction – time varying magnetic flux produces circulating $\vec{E}$-field

3) Gauss Law for magnetic fields – no magnetic monopoles. Magnetic field lines are continuous, they either close upon themselves or go off to infinity, they cannot begin nor end at any point.

4) Amperes Law + Maxwell's correction – electric current is a source for circulating $\vec{B}$-field; so is a time varying $\vec{E}$-field. Maxwell's correction is necessary to have charge conservation and to give electromagnetic waves.