$\Rightarrow$ tangential component of $\vec{E}$ is continuous

combine above to write $\boxed{\vec{E}^{top} - \vec{E}^{bottom} = 4\pi\sigma(\vec{r})\,\hat{n}}$

iii) $\vec{E} = -\vec{\nabla}\phi \Rightarrow \phi(r_2) - \phi(r_1) = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{E}$

Take $\vec{r}_2$ just above $\vec{r}$ on surface
$\vec{r}_1$ just below $\vec{r}$ on surface $\Big\}\ d\vec{l} \to 0$

Since $\vec{E}$ is finite $\Rightarrow \int d\vec{l}\cdot\vec{E} \to 0$

$\Rightarrow \boxed{\phi^{top} = \phi^{bottom}}$

potential ϕ is continuous at surface charge layer

can rewrite (i) as

$$\left(-\vec{\nabla}\phi^{top} + \vec{\nabla}\phi^{bottom}\right)\cdot\hat{n} = 4\pi\sigma$$

$$\boxed{-\frac{\partial\phi^{top}}{\partial n} + \frac{\partial\phi^{bottom}}{\partial n} = 4\pi\sigma}$$

directional derivative of ϕ in direction $\hat{n}$

discontinuity in normal derivative of ϕ at surface

Apply to conducting sphere

ϕ continuous $\Rightarrow \phi^{in}(R) = \phi^{out}(R)$

$$C_1^{in} = \frac{C_0^{out}}{R}$$

only one unknown C_0^{out}

normal derivative of ϕ is discontinuous

$$-\frac{\partial \phi^{top}}{\partial n} + \frac{\partial \phi^{bottom}}{\partial n} = 4\pi\sigma$$

here $\hat{n} = \hat{r}$ the radial direction

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr}\right]_{r=R} = 4\pi\sigma$$

but $\dfrac{d\phi^{in}}{dr} = 0$ as $\phi^{in} = \text{constant}$

$$-\left.\frac{d\phi^{out}}{dr}\right|_{r=R} = 4\pi\sigma$$

charge q is uniformly distributed on surface at R

$$-\frac{d}{dr}\left(\frac{C_0^{out}}{r}\right)_{r=R} = \frac{C_0^{out}}{R^2} = 4\pi\sigma = 4\pi\left(\frac{q}{4\pi R^2}\right) = \frac{q}{R^2}$$

$$\Rightarrow C_0^{out} = q, \quad C^{in} = \frac{C^{out}}{R} = \frac{q}{R}$$

$$\phi(r) = \begin{cases} \dfrac{q}{R} & r < R \quad \text{inside} \\ \dfrac{q}{r} & r > R \quad \text{outside} \end{cases}$$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi = -\frac{d\phi}{dr} = \begin{cases} 0 & r<R \quad \text{inside} \\ \dfrac{q}{r^2} & r>R \quad \text{outside} \end{cases}$$

we get familiar Coulomb solution!

Summary We can view the preceeding solution for ϕ^{out} as solving Laplace's equ $\nabla^2\phi = 0$ subject to a specified boundary condition on the normal derivative of ϕ at the boundary $r=R$ of the "outside" region of the system.

Alternate problem:

Another physical situation would be to connect conducting sphere to a battery that charges the sphere to a fixed voltage ϕ_0 (stat volts!) with respect to ground $\phi=0$ at $r\to\infty$.

As before, outside the sphere $\phi = \frac{C_0}{r}$
Now the boundary condition is to specify the value of ϕ on the boundary of the outside region, ie

$$\phi(R) = \phi_0$$

$$\Rightarrow \frac{C_0}{R} = \phi_0 , \quad C_0 = \phi_0 R$$

$$\phi(r) = \phi_0 \frac{R}{r}$$

(from preceeding solution, we know that charging the sphere to voltage ϕ_0 (statvolts) induces a net charge $q = \phi_0 R$ on it)

These two versions of the conducting sphere problem are examples of a more general <u>boundary value problem</u>

Solve $\nabla^2 \phi = 0$ in a given region of space subject to one of the following two types of boundary conditions on the bounding surfaces of the region

i) Neumann boundary condition

$\frac{\partial \phi}{\partial n}$ - normal derivative of ϕ is specified on the bounding surface

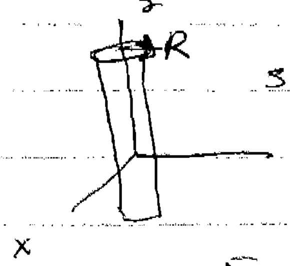
ii) Dirichlet boundary condition

ϕ - value of ϕ is specified on the bounding surfaces

If the bounding surfaces consist of <u>disjoint</u> pieces, it is possible to specify either (i) or (ii) on each piece separately to get a mixed boundary value problem.

Some more problems

infinite conducting wire of radius R with line charge density λ = charge per unit length



surface charge $\sigma = \frac{\lambda}{2\pi R}$

Expect cylindrical symmetry $\Rightarrow \phi$ depends only on cylindrical coord r.

$\nabla^2\phi = 0$ for $r > R$, $r < R$

use ∇^2 in cylindrical coords – only radial term non vanishing

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

$$r\frac{d\phi}{dr} = C_0 \quad \text{constant}$$

$$\frac{d\phi}{dr} = \frac{C_0}{r}$$

$$\phi(r) = C_0 \ln r + C_1 \quad \text{const}$$

note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

one needs to fix zero of ϕ at some other radius. a convenient choice is $r = R$, but any other choice could also be made.

$$\phi^{out} = C_0^{out} \ln r + C_1^{out}$$
$$\phi^{in} = C_0^{in} \ln r + C_1^{in}$$

ϕ^{in} = const in conductor $\Rightarrow C_0^{in} = 0$

or ϕ^{in} should not diverge as $r \to 0 \Rightarrow C_0^{in} = 0$

So $\phi^{in} = C_1^{in}$ constant

boundary condition at $r = R$

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr} \right]_{r=R} = 4\pi\sigma$$

$$\Rightarrow \quad -\frac{C_0^{out}}{R} = 4\pi\sigma = 4\pi\left(\frac{\lambda}{2\pi R}\right) = \frac{2\lambda}{R}$$

$$C_0^{out} = -2\lambda$$

$$\phi^{out}(r) = -2\lambda \ln r + C_1^{out}$$

continuity of ϕ

$$\phi^{in}(R) = \phi^{out}(R) \quad \Rightarrow \quad C_1^{in} = -2\lambda \ln R + C_1^{out}$$

Remaining const C_1^{out} is not too important as it is just a common additive constant to both ϕ^{in} and ϕ^{out} $\Rightarrow$ does not change $\vec{E} = -\vec{\nabla}\phi$.

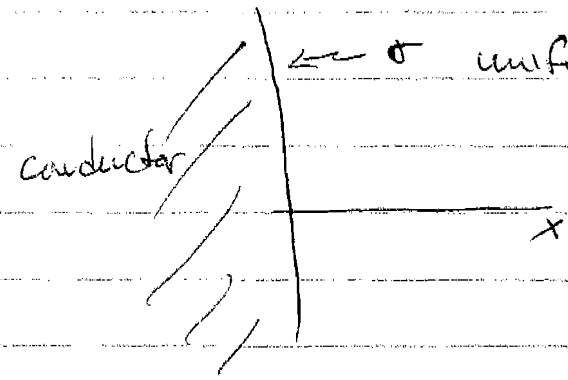
If use the condition $\phi(R) = 0$ then we can solve for C_1^{out}.

$$0 = -2\lambda \ln R + c_1^{out} \Rightarrow c_1^{out} = 2\lambda \ln R$$

$$\Rightarrow \phi(r) = \begin{cases} -2\lambda \ln(r/R) & r > R \\ 0 & r < R \end{cases}$$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \frac{2\lambda}{r}\,\hat{r} & r > R \\ 0 & r < R \end{cases}$$

infinite conducting half space



$\leftarrow \sigma$ uniform surface charge density

expect ϕ depends only on x

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = 0$$

$$\rightarrow \begin{cases} \phi^>(x) = c_0^> x + c_1^> & x > 0 \\ \phi^<(x) = c_0^< x + c_1^< & x < 0 \end{cases}$$

for $x < 0$, $\phi = \text{const}$ in conductor $\Rightarrow c_0^< = 0$

at $x = 0$, ϕ continuous $\Rightarrow \phi^<(0) = \phi^>(0)$

$$c_1^< = c_1^>$$

$\frac{d\phi}{dx}$ discontinuous $\Rightarrow$

$$-\left.\frac{d\phi^>}{dx}\right|_{x=0} = 4\pi\sigma$$

$$c_0^> = -4\pi\sigma$$

$$\Rightarrow \phi(x) = \begin{cases} -4\pi\sigma x + c_1^> & x > 0 \\ c_1^> & x < 0 \end{cases}$$

const $c_1^>$ does not change value of $\vec{E}$

as for the wire, we cannot choose $\phi \to 0$ as $x \to \infty$.
~~we can set $\phi = 0$ at~~

$$-\vec{\nabla}\phi = \vec{E} = \begin{cases} 4\pi\sigma\,\hat{x} & x > 0 \\ 0 & x < 0 \end{cases}$$

<u>infinite charged plane</u>

similar to previous problem, but now no conductor at $x < 0$, just free space on both sides of the charged plane at $x = 0$.

~~expect ϕ to depend only on $|x|$ by symmetry~~

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = 0 \quad \Rightarrow \quad \begin{array}{ll} \phi^> = c_0^> x + c_1^> & x > 0 \\ \phi^< = c_0^< x + c_1^< & x < 0 \end{array}$$

continuity of ϕ at $x = 0$

$$\to \quad \phi^>(0) = \phi^<(0) \quad \Rightarrow \quad c_1^> = c_1^<$$

discontinuity of $d\phi/dx$ at $x = 0$

$$-\frac{d\phi^>}{dx} + \frac{d\phi^<}{dx} = 4\pi\sigma$$

$$-c_0^> + c_0^< = 4\pi\sigma$$

Define $\bar{c}_0 = \dfrac{c_0^> + c_0^<}{2}$

Then we can write

$$C_0^< = \bar{C}_0 + 2\pi\sigma$$
$$C_0^> = \bar{C}_0 - 2\pi\sigma$$

$$\phi = \begin{cases} -2\pi\sigma x + \bar{C}_0 x + C_1^> & x > 0 \\ 2\pi\sigma x + \bar{C}_0 x + C_1^> & x < 0 \end{cases}$$

$$\Rightarrow -\frac{d\phi}{dx} = \vec{E} = \begin{cases} (2\pi\sigma - \bar{C}_0)\,\hat{x} & x > 0 \\ (-2\pi\sigma - \bar{C}_0)\,\hat{x} & x < 0 \end{cases}$$

Const $C_1^>$ does not effect $\vec{E}$ - additive const to ϕ

$\bar{C}_0$ represents const uniform electric field $-\bar{C}_0\hat{x}$, that exists independently of the charged surface

If we assumed that all $\vec{E}$ fields are just those arising from the plane, then we can set $\bar{C}_0 = 0$. Equivalently, if the plane is the only source of $\vec{E}$, then we expect ϕ depends only on $|x|$ by symmetry.
$\Rightarrow$ $C_0^< = -C_0^>$ and again $\bar{C}_0 = 0$. In this case

$$\phi(x) = \begin{cases} -2\pi\sigma x & x > 0 \\ 2\pi\sigma x & x < 0 \end{cases}$$

(we also set $C_1^> = 0$ here corresponding to $\phi(0) = 0$)

$$\vec{E}(x) = \begin{cases} 2\pi\sigma\,\hat{x} & x > 0 \\ -2\pi\sigma\,\hat{x} & x < 0 \end{cases}$$

$\vec{E}$ is constant ~~by~~ but oppositely directed on either side of the charged plane

Green's Theorem, Uniqueness, Green function - part II

We want to show that the boundary value problem we described is well posed - ie there is a unique solution. We start by deriving Green's Theorems

Consider $\int_V d^3r\, \vec{\nabla}\cdot\vec{A} = \oint_S da\, \hat{n}\cdot\vec{A}$ Gauss theorem

let $\vec{A} = \phi\vec{\nabla}\psi$ ϕ, ψ any two scalar functions

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{A} = \phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$$

$$\phi\,\vec{\nabla}\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$

$$\Rightarrow \int_V d^3r\left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\,\phi\frac{\partial\psi}{\partial n}$$

Green's 1st - identity

let $\phi \leftrightarrow \psi$

$$\int_V d^3r\left(\psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\,\psi\frac{\partial\phi}{\partial n}$$

subtract

$$\int_V d^3r\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S da\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)$$

Green's 2nd identity

Apply Greens 2nd identity with $\psi = \frac{1}{|\vec{r}-\vec{r}'|}$ → $\vec{r}'$ is integration variable, ϕ is the scalar potential with $\nabla^2\phi = -4\pi\rho$. Use $\nabla^2\psi = \nabla'^2\psi = -4\pi\,\delta(\vec{r}-\vec{r}')$

$$\int_V d^3r'\left[\phi(r')\left[-4\pi\,\delta(r-r')\right] - \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\left(-4\pi\rho(\vec{r}')\right)\right]$$

$$= \oint_S da'\left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'}\right]$$

If $\vec{r}$ lies within the volume V, then

$$(*) \qquad \phi(\vec{r}) = \int_V d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

Note: if $\vec{r}$ lies outside the volume V, then

$$0 = \int_V d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint \frac{da'}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]$$

potential from a surface charge density $\sigma = \frac{1}{4\pi}\frac{\partial\phi}{\partial n'}$

potential from a surface dipole layer of dipole strength density $\frac{\phi}{4\pi}$

From (*), if $S \to \infty$ and $E \sim \frac{\partial\phi}{\partial n} \to 0$ faster than $\frac{1}{r}$, then the surface integral vanishes and we recover Coulombs law $\phi(\vec{r}) = \int_V d^3r' \, \rho(\vec{r}')/|\vec{r}-\vec{r}'|$

(*) gives the generalization of Coulombs law to a system with a finite boundary

For a charge free volume V, ie $\rho(r) = 0$ in V, the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify both ϕ and $\frac{\partial\phi}{\partial n}$ on the boundary surface since the resulting ϕ from (*) would not in general obey Laplace's equation $\nabla^2\phi = 0$.

Specifying both ϕ and $\frac{\partial\phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's eqn, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

<u>Uniqueness</u>

If we have a system of charges in vol V, and either the potential ϕ, or its normal derivative $\frac{\partial\phi}{\partial n}$, is specified on the surfaces of V, then there is a unique solution to Poisson's equation inside V. Specifying ϕ is known as Dirichlet boundary conditions. Specifying $\frac{\partial\phi}{\partial n}$ is known as Neumann boundary conditions.

<u>proof</u>: Suppose we had two solutions ϕ_1 and ϕ_2, <u>both</u> with $-\nabla^2\phi = 4\pi\rho$ inside V, and obeying specified b.c. on surface of V.

Define $u = \phi_2 - \phi_1 \rightarrow \nabla^2 u = 0$ inside V

and $u = 0$ on surface S - for Dirichlet b.c.
or $\frac{\partial u}{\partial n} = 0$ on surface S - for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r \, (\underbrace{u\nabla^2 u}_{=0 \text{ as } \nabla^2 u = 0} + \vec{\nabla}u\cdot\vec{\nabla}u) = \oint_S \underbrace{da \, u\frac{\partial u}{\partial n}}_{=0 \text{ as } u \text{ or } \frac{\partial u}{\partial n} = 0}$$

$$\Rightarrow \int_V d^3r \, |\vec{\nabla} U|^2 = 0 \quad \Rightarrow \vec{\nabla} U = 0$$
$$\Rightarrow U = \text{const}$$

For Dirichlet b.c., $U = 0$ on surface S, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., ϕ_1 and ϕ_2 differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $\vec{E}_2 = -\vec{\nabla}\phi_2$ are the same.

If boundary ~~solution~~ surface S consists of several disjoint pieces, then solution is unique if specify ϕ on some pieces and $\frac{\partial \phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with <u>both</u> ϕ <u>and</u> $\frac{\partial \phi}{\partial n}$ specified on the same surface S (Cauchy b.c.) does not in general exist, since specifying either ϕ <u>or</u> $\frac{\partial \phi}{\partial n}$ alone is enough to give a unique solution.