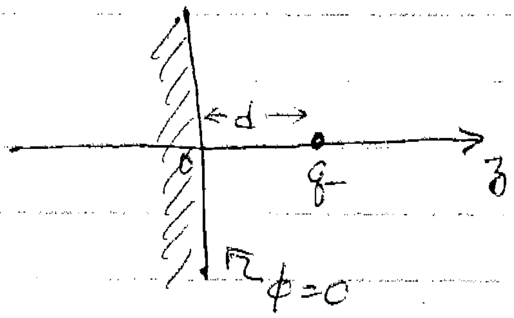
Image Charge method

For simple geometries, can try to obtain G_D or G_N by placing a set of "image charges" outside the volume of interest V, ie on the "other side" of the system boundary surface S. Because these image charges are outside V, their contrib to the potential inside V obeys $\nabla^2\phi^{\text{image}} = 0$, as necessary. Choose location of image charges so that total ϕ has desired boundary condition.

1) Charge in front of infinite grounded plane



want $\nabla^2\phi = -4\pi q\,\delta(x)\delta(y)\delta(z-d)$

$\phi = 0$ for $z = 0$

If we find a solution to above it is the unique solution

Solution - put fictitious image charge $-q$ at $z = -d$

ϕ is Coulomb potential from the real charge + the image

$$\phi(\vec{r}) = \underbrace{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\text{real charge}} + \underbrace{\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}}_{\text{image charge}}$$

above satisfies $\phi(x,y,0) = 0$ as required

Also, $\nabla^2\phi = -4\pi q\,\delta(\vec{r}-d\hat{z}) + 4\pi q\,\delta(\vec{r}+d\hat{z})$

$= -4\pi q\,\delta(\vec{r}-d\hat{z})$ for region $z > 0$

Can now find $\vec{E}$ for $z > 0$

$$\vec{E} = -\vec{\nabla}\phi$$

In particular $E_z = -\frac{\partial \phi}{\partial z} = +q\left[\left(\frac{1}{2}\right)\frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}}\right.$

$$\left. - \left(\frac{1}{2}\right)\frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

$$E_z = q\left[\frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

We can use above to compute the surface charge density $\sigma(x,y)$ induced on the surface of the conducting plane. At conductor surface

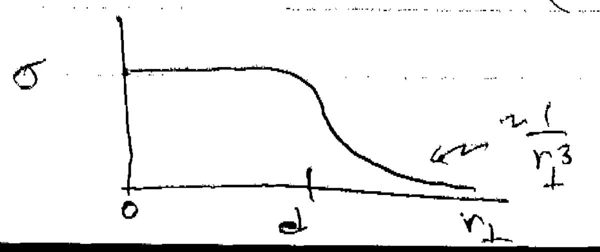
$$-\frac{\partial \phi}{\partial n} = 4\pi\sigma$$

$$\Rightarrow \quad \sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial z} = \frac{1}{4\pi}E_z(x,y,z=0)$$

$$\sigma(x,y) = \frac{q}{4\pi}\left[\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$= -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}} = \frac{-qd}{2\pi\,(r_\perp^2+d^2)^{3/2}}$$

$$r_\perp = \sqrt{x^2+y^2}$$



Total induced charge is

$$q^{induced} = \int_{-\infty}^{\infty} dx\,dy\ \sigma(x,y)$$

$$= 2\pi \int_0^{\infty} dr_\perp\, r_\perp\ \sigma(r_\perp)$$

$$= 2\pi \int_0^{\infty} dr_\perp\ \frac{r_\perp\ (-qd)}{2\pi\left(r_\perp^2+d^2\right)^{3/2}}$$

$$= -qd \left[\frac{-1}{(r_\perp^2+d^2)^{1/2}}\right]_0^{\infty}$$

$$= -qd\left[0 - \frac{-1}{d}\right]$$

$$q^{induced} = -q \qquad \text{induced charge = image charge}$$

Force on charge q in front of conducting plane is due to the induced σ. The E field of this σ is, for $z>0$, the same as the E field of the image charge.

$$\Rightarrow \vec{F} = \frac{-q^2}{(2d)^2}\hat{z} = \frac{-q^2}{4d^2}\hat{z} \qquad \underline{\text{attractive}}$$

Work done to move q into position from infinity is

$$W = -\int_{\infty}^{d} d\vec{\ell}\cdot\vec{F} = -\int_{\infty}^{d} dz\ F_z$$

we must oppose electrostatic force $\vec{F}$

$$W = \int_d^{\infty} dz \left(\frac{-q^2}{4z^2} \right) = \frac{-q^2}{4d}$$

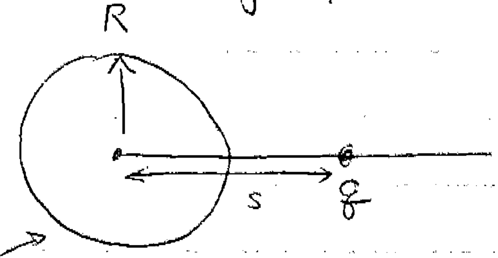
$W < 0 \Rightarrow$ energy released

<u>Note:</u> W above is <u>not</u> the electrostatic energy that would be present if the image charge were real

i.e. it is <u>not</u> $q\phi^{\text{image}}(\vec{r} = d\hat{z}) = \dfrac{-q^2}{2d}$

One way to see why is to note that as q is moved quasistatically in towards the conducting plane, the image charge also must be moving to stay equidistant on the opposite side,

2) point charge in front of a grounded ($\phi=0$) conducting sphere.

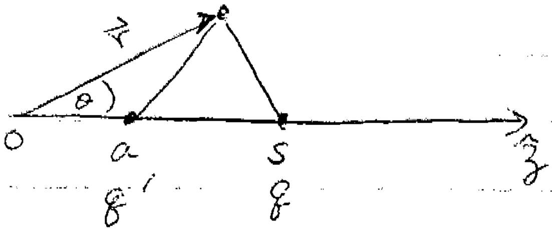


grounded
$\phi=0$

charge q placed a distance s from center of grounded conducting sphere of radius R

place image charge q' inside sphere so that the combined Φ from q and q' vanishes on surface of sphere.

By symmetry, q' should lie on the same radial line as q does. Call the distance of q' from the origin "a"



potential at position $\vec{r}$ is

$$\Phi(\vec{r}) = \frac{q}{|\vec{r} - s\hat{z}|} + \frac{q'}{|\vec{r} - a\hat{z}|}$$

$$= \frac{q}{(r^2+s^2-2sr\cos\theta)^{1/2}} + \frac{q'}{(r^2+a^2-2ra\cos\theta)^{1/2}}$$

Can we choose q' and a so that $\Phi(R,\theta)=0$ for all θ?

$$\phi(R,\theta) = \frac{q}{(R^2+s^2-2sR\cos\theta)^{1/2}} + \frac{q'}{(R^2+a^2-2aR\cos\theta)^{1/2}}$$

make denominators look alike

$$R^2+a^2-2aR\cos\theta = \frac{a}{s}\left(\frac{s}{a}R^2 + sa - 2sR\cos\theta\right)$$

if choose $sa = R^2$, ie $\boxed{a = R^2/s}$, then $\frac{sR^2}{a} = s^2$
and then the denominator of the 2nd term is

$$\left[\frac{R^2}{s^2}(s^2+R^2-2sR\cos\theta)\right]^{1/2} = \frac{R}{s}\left[s^2+R^2-2sR\cos\theta\right]^{1/2}$$

$$\Rightarrow \quad \phi(R,\theta) = \frac{q}{(R^2+s^2-2sR\cos\theta)^{1/2}} + \frac{q'(s/R)}{(R^2+s^2-2sR\cos\theta)^{1/2}}$$

So choose $q'(s/R) = -q \rightarrow \boxed{q' = -qR/s}$
to get $\phi(R,\theta)=0$

Solution is

$$\phi(r,\theta) = \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{qR/s}{\left(r^2+\frac{R^4}{s^2}-2r\frac{R^2}{s}\cos\theta\right)^{1/2}}$$

$$= \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q}{\left(\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta\right)^{1/2}}$$

Can get induced surface charge on sphere by

$4\pi\sigma = \vec{E}\cdot\hat{r} = -\left.\frac{\partial\phi}{\partial r}\right|_{r=R}$ see Jackson Eq (2.5) for result

$$\sigma(\theta) = -\frac{q}{4\pi}\frac{1}{RS}\frac{1-(R/s)^2}{\left(1+(R/s)^2-2(R/s)\cos\theta\right)^{3/2}}$$

$\sigma(\theta)$ is greatest at $\theta=0$, as one should expect

Can integrate $\sigma(\theta)$ to get total induced charge. One finds

$$2\pi\int_0^\pi d\theta \sin\theta\, R^2\,\sigma(\theta) = q' = -qR/s$$

In general, total induced charge = sum of all image charges

Force of attraction of charge to sphere

Force on q is due to electric field from induced charge σ which is the same as the electric field from the image charge q'.

$$\vec{F} = \frac{qq'\,\hat{z}}{(s-a)^2} = \frac{-q^2(R/s)\,\hat{z}}{(s-R^2/s)^2} = \frac{-q^2Rs}{(s^2-R^2)^2}\,\hat{z}$$

Close to the surface of the sphere, $s\approx R$, so write $s=R+d$ where $d\ll R$. Then

$$\vec{F} = \frac{-q^2Rs}{(s-R)^2(s+R)^2} = -\frac{q^2R(R+d)}{d^2\,(2R+d)^2} \simeq \frac{-q^2}{4d^2}$$

get same result as for infinite flat grounded plane. When q is so close to surface that $d\ll R$, the charge does not "see" the curvature of the surface far from the surface, $s \gg R$

$$\vec{F} = \frac{q q' \hat{z}}{(s-a)^2} = \frac{-q^2 R s}{(s^2 - R^2)^2} \hat{z} \simeq -\frac{q^2 R}{s^3} \hat{z}$$

$F \sim \frac{1}{s^3}$ very different from flat plane
also different from point charge

Note: In preceding two problems, what we found was a ϕ such that $\nabla^2 \left(\frac{\phi}{q}\right) = -4\pi\delta(\vec{r}-\vec{r}_0)$, for a charge at $\vec{r}_0$, and $\phi = 0$ on the boundary. Such a ϕ is nothing more than G_D the corresponding Green function for Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we have a sphere with fixed net charge Q.

We want to add new image charge to represent this case. If we put $q' = -qR/s$ at $a = R/s$ as before, the boundary condition of ϕ = const on surface $r = R$ is met, but the net charge on the sphere is q' (the induced charge) not the desired Q. We therefore need to add new image charge(s) of total charge $Q - q'$ (so total image charge is Q) in such a way that we keep ϕ constant on the surface of the sphere. The way to do this is to put $Q - q'$ at the origin!

Solution is

$$\phi(r,\theta) = \frac{Q + qR/s}{r} + \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q}{\left(\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta\right)^{1/2}}$$

The force on the charge q is due to the $\vec{E}$ field of the images

$$\vec{F} = F\hat{z} = \frac{q(Q+qR/s)\hat{z}}{s^2} + \frac{qq'\hat{z}}{(s-a)^2}$$

$$F = \frac{qQ}{s^2} + \frac{q^2R/s}{s^2} - \frac{q^2R/s}{(s-R^2/s)^2}$$

$$= \frac{qQ}{s^2} + q^2R\left[\frac{1}{s^3} - \frac{1}{s^3(1-R^2/s^2)^2}\right]$$

$$= \frac{qQ}{s^2} + \frac{q^2R}{s^3}\left[1 - \frac{1}{(1-R^2/s^2)^2}\right]$$

$$F = \frac{qQ}{s^2} - q^2\frac{R^3}{s}\frac{2-R^2/s^2}{(s^2-R^2)^2}$$

For large $s \gg R$ far from surface

$$F \simeq \frac{qQ}{s^2} - \frac{2q^2R^3}{s^5}$$

leading term is just Coulomb force between q and Q at origin

for $Q>0$, F is always repulsive for large enough s.

For $s = R+d$, $d \ll R$ close to surface

$$F = \frac{qQ}{(R+d)^2} - \frac{q^2 R^3}{(R+d)} \frac{2 - \frac{R^2}{(R+d)^2}}{(R^2+d^2+2Rd-R^2)^2}$$

$$\simeq \frac{qQ}{R^2} - \frac{q^2 R^3}{R} \frac{(2-1)}{4R^2 d^2}$$

$$F \simeq \frac{qQ}{R^2} - \frac{q^2}{4d^2} \simeq -\frac{q^2}{4d^2} \quad \text{for } d \text{ small enough}$$

F is always attractive for small enough d, and is equal to the force in front of a grounded plane, no matter what is the value of Q! This is because the image charge q' lies so much closer to q than does the $Q-q'$ at the origin, that it dominates the force.

The cross over from attractive to repulsive occurs at a distance s that depends on Q. This distance is given by

$$\frac{Q}{q} = \frac{R^3 s\,(2 - R^2/s^2)}{(s^2 - R^2)^2} = \left(\frac{R}{s}\right)^3 \frac{2-(R/s)^2}{[1-(R/s)^2]^2}$$

let $x = R/s \in (0,1)$

$$\frac{Q}{q} = x^3 \frac{(2-x^2)}{(1-x^2)^2}$$

gives 5th order polynomial in x
no analytic solution
can solve graphically

For $\frac{Q}{q} = 1$, cross over is at $\frac{R}{s} = 0.62$

$$s = 1.6 R$$

$\frac{Q}{q} = 0.1$ cross over is at $\frac{R}{s} \simeq 0.36$

$$s = 2.8 R$$