

## Separation of Variables

If the system has a rectangular boundary, and contains no charge, we can look for solutions to  $\nabla^2 \phi = 0$  of the form

$$\phi(\vec{r}) = X(x) Y(y) Z(z) \quad \text{product of three functions each of one variable only}$$

$$\nabla^2 \phi = 0 \Rightarrow \frac{1}{\phi} \nabla^2 \phi = 0$$

$$\Rightarrow \frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z}{dz^2} = 0$$

The only way this can happen for all values of  $x, y, z$  is if each of the three terms is a constant, call them  $a^2, b^2, c^2$ .

$$\frac{1}{X} \frac{d^2 X}{dx^2} = a^2 \Rightarrow X(x) = A_1 e^{-ax} + A_2 e^{ax}$$

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = b^2 \Rightarrow Y(y) = B_1 e^{-by} + B_2 e^{by}$$

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = c^2 \Rightarrow Z(z) = C_1 e^{-cz} + C_2 e^{cz}$$

$$\text{with } a^2 + b^2 + c^2 = 0$$

$\Rightarrow$  at least one of the  $a^2, b^2, c^2$  is  $< 0$

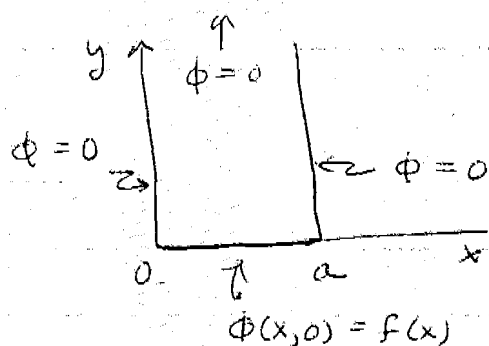
$\Rightarrow$  at least one of the  $a, b, c$  is imaginary.

Above is one particular solution. But there are many solutions, each with different  $a, b, c$ , but all obeying the constraint  $a^2 + b^2 + c^2 = 0$ . The General solution is a superposition of these

$$\phi(x, y, z) = \sum_i (A_{1i} e^{-a_i x} + A_{2i} e^{a_i x}) (B_{1i} e^{-b_i y} + B_{2i} e^{b_i y}) (C_{1i} e^{-c_i z} + C_{2i} e^{c_i z})$$

Example  $a_i^2 + b_i^2 + c_i^2 = 0$

Consider a channel shaped as below - infinite along  $z$



$$\phi(0, y) = 0$$

$$\phi(a, y) = 0$$

$$\phi(x, y) = 0 \text{ as } y \rightarrow \infty$$

$$\phi(x, 0) = f(x) \text{ specified function}$$

solution is independent of  $z \Rightarrow$

$$\phi(x, y) = \sum_i (A_{1i} e^{-a_i x} + A_{2i} e^{a_i x}) (B_{1i} e^{-b_i y} + B_{2i} e^{b_i y})$$

$$a_i^2 + b_i^2 = 0$$

we will see that the correct thing is to choose a imaginary

$$\text{let } a_i = i\alpha_i$$

$$b_i = \alpha_i$$

$$\phi(x, y) = \sum_i (A_i \cos \alpha_i x + B_i \sin \alpha_i x) (C_i e^{-\alpha_i y} + D_i e^{\alpha_i y})$$

where  $A_i = (A_{1i} + A_{2i})$

$$B_i = i(A_{1i} - A_{2i})$$

$$C_i = B_{1i}$$

$$D_i = B_{2i}$$

Now  $\phi(x, y) \rightarrow 0$  as  $y \rightarrow \infty$  for all  $x \Rightarrow \boxed{D_z = 0}$

$$\Rightarrow \phi(x, y) = \sum_i \left[ A_i' \cos \alpha_i x + B_i' \sin \alpha_i x \right] e^{-\alpha_i y}$$

$$\text{where } A_i' = A_i C_i, \quad B_i' = B_i C_i$$

$$\phi(0, y) = 0 \Rightarrow \sum_i A_i' e^{-\alpha_i y} = 0 \text{ all } y \Rightarrow \boxed{A_i' = 0}$$

$$\Rightarrow \phi(x, y) = \sum_i B_i' \sin(\alpha_i x) e^{-\alpha_i y}$$

$$\phi(a, y) = 0 \Rightarrow \sum_i B_i' \sin(\alpha_i a) e^{-\alpha_i y} = 0 \text{ all } y$$

$$\Rightarrow \sin(\alpha_i a) = 0 \text{ or } \alpha_i a = n\pi$$

$$\alpha_i = \frac{n\pi}{a} \text{ integer } n \geq 1$$

$$\Rightarrow \boxed{\phi(x, y) = \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}}$$

Finally

$$\phi(x, 0) = f(x) \Rightarrow \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) = f(x)$$

This is just the Fourier series for  $f(x)$ !

$$\boxed{B_n' = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx}$$

We have thus determined all unknown coefficients and found the solution!

See Jackson 2-8 if  
Fourier series needs review

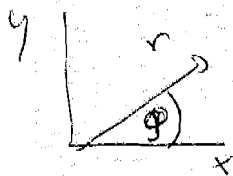
Recall orthogonality :  $\frac{2}{a} \int_0^a dx \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

For  $f(x) = \phi_0$  a constant,

$$\begin{aligned} B_n' &= \frac{2}{a} \phi_0 \int_0^a dx \sin\left(\frac{n\pi x}{a}\right) = \frac{2\phi_0}{a} \left[ -\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right) \right]_0^a \\ &= \frac{2\phi_0}{n\pi} (1 - \cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{4\phi_0}{n\pi} & n \text{ odd} \end{cases} \end{aligned}$$

Polar Coordinates - still translationally invariant along  $z$  - so two dimensional

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \varphi^2} = 0$$



assume  $\phi(r, \varphi) = R(r) \Phi(\varphi)$

$$\frac{r^2 \nabla^2 \phi}{\phi} = \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = 0$$

each term must be a constant

$$\Rightarrow \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = \nu^2, \quad \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = -\nu^2$$

$$\text{Solutions are } \left. \begin{aligned} R(r) &= ar^\nu + br^{-\nu} \\ \Phi(\varphi) &= A \cos(\nu\varphi) + B \sin(\nu\varphi) \end{aligned} \right\} \nu \neq 0$$

$$\left. \begin{aligned} R(r) &= a_0 + b_0 \ln r \\ \Phi(\varphi) &= A_0 + B_0 \varphi \end{aligned} \right\} \nu = 0$$

~~$$\phi(r, \varphi) = (a_0 + b_0 \ln r) (A_0 + B_0 \varphi) + \sum_{n=1}^{\infty} r^n (A_n \cos(n\varphi) + B_n \sin(n\varphi)) + r^{-n} (C_n \cos(n\varphi) + D_n \sin(n\varphi))$$~~

If  $\varphi$  can take its entire range from 0 to  $2\pi$  then (such as problem in which  $\phi$  is specified on the surface of a cylinder) then periodicity in  $\varphi \rightarrow \varphi + 2\pi$  requires  $B_0 = 0$  and  $\nu = \text{integer } n$

$$\phi = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[ r^n (A_n \cos(n\varphi) + B_n \sin(n\varphi)) + r^{-n} (C_n \cos(n\varphi) + D_n \sin(n\varphi)) \right]$$

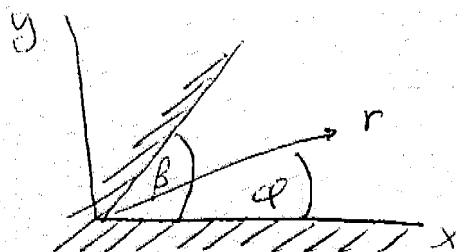
or reparameterizing

$$\phi(r, \varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[ a_n r^n \sin(n\varphi + \alpha_n) + b_n r^{-n} \sin(n\varphi + \beta_n) \right]$$

If the region where there is no charge includes  $r=0$ , then all  $b_n = 0$  since  $\phi$  should not diverge at the origin.

If  $r=0$  is excluded from the region, then the  $b_n$  need not be zero. The case  $b_0 \neq 0$  corresponds to a line charge  $\lambda$  along the  $z$  axis.

Consider the case where  $\varphi$  has a restricted range, for example a wedge shaped opening of angle  $\beta$



shaded region is conductor

$$0 \leq \varphi \leq \beta$$

$\phi$  is constant in conductor  
 $\Rightarrow$  boundary conditions

$$\begin{cases} \phi(r, 0) = \phi_0 \\ \phi(r, \beta) = \phi_0 \end{cases}$$

The general solution is the linear combination

$$\phi(r, \varphi) = (a_0 + b_0 \ln r)(A_0 + B_0 \varphi)$$

$$+ \sum_{v \geq 1} (a_v r^v + b_v r^{-v})(A_v \cos(v\varphi) + B_v \sin(v\varphi))$$

- ① The condition  $\phi(r, 0) = \phi_0$  a constant indep of  $r$  then requires

$$b_0 = 0, \quad A_v = 0 \quad \text{all } v$$

So

$$\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{v>0} (a_v r^v + b_v r^{-v}) B_v \sin(v\varphi)$$

- ② Since  $\phi$  should be continuous as one approaches the conducting surface, and  $\phi = \phi_0$  is a finite constant on the conducting surface, then  $\phi$  cannot diverge as one approaches the origin  $r=0$  along any fixed angle  $\varphi$ . This requires

$$b_v = 0 \quad \text{all } v$$

So

$$\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{v>0} a_v r^v \sin(v\varphi)$$

- ③ The condition  $\phi(r, \beta) = \phi_0$  a constant independent of  $r$  then requires

$$\sin(v\beta) = 0 \Rightarrow v = \frac{n\pi}{\beta}, \quad n \text{ integer } \geq 1$$

So

$$\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\varphi}{\beta}\right)$$

- ④ as  $\phi$  must approach the constant  $\phi_0$  as  $r \rightarrow 0$  along any fixed angle  $\varphi$ , we therefore must have

$$B_0 = 0, \quad a_0 A_0 = \phi_0$$

So finally we have

$$\phi(r, \varphi) = \phi_0 + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\varphi}{\beta}\right)$$

We still have all the unknown  $a_n$ ! These depend on how  $\phi(r, \varphi)$  behaves as  $r \rightarrow \infty$  (we can't make the choice here that  $\phi \rightarrow 0$  as  $r \rightarrow \infty$ ) - this is additional information that must be specified to find the complete solution.

Nevertheless we can still get very interesting information near the origin at small  $r$ . In this case, the leading term in the above series expansion for  $\phi$  is the  $n=1$  term, as it vanishes most slowly as  $r \rightarrow 0$ .

$$\phi(r, \varphi) \simeq \phi_0 + a_1 r^{\frac{\pi}{\beta}} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

The electric field is

$$E_r(r, \varphi) = -\frac{\partial \phi}{\partial r} = -\frac{\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

$$E_\varphi(r, \varphi) = -\frac{1}{r} \frac{\partial \phi}{\partial \varphi} = -\frac{\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \cos\left(\frac{\pi\varphi}{\beta}\right)$$

$$\Rightarrow \boxed{E \sim r^{\frac{\pi}{\beta}-1}}$$

Induced surface charge given by  $4\pi\sigma = \vec{E} \cdot \hat{n}$

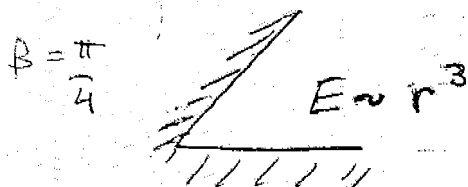
for surface at  $\varphi=0$ ,  $\hat{n} = \hat{\varphi}$   
 for surface at  $\varphi=\beta$ ,  $\hat{n} = -\hat{\varphi}$

$$\sigma(r, \varphi=0) = \frac{E_{\varphi}(r, 0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1}$$

$$\sigma(r, \varphi=\beta) = -\frac{E_{\varphi}(r, \beta)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1}$$

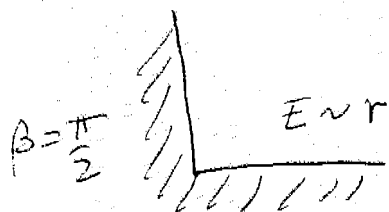
For  $\frac{\pi}{\beta} > 1$ , i.e.  $\beta < \pi$ ,  $\vec{E}$  and  $\sigma$  vanish as approach the origin.

For  $\frac{\pi}{\beta} < 1$ , i.e.  $\beta > \pi$ ,  $\vec{E}$  and  $\sigma$  diverge as approach the origin.



$\beta = \pi$

$\vec{E} \sim \text{const}$



$\beta = \frac{3\pi}{2}$

$E \sim r^{-1/3}$

$\beta = 2\pi$

$E \sim r^{-1/2}$

$E$  diverges at an "external" corner

$E$  vanishes at an "internal" corner

Remember, the above examples all had translational symmetry along  $z$ , so the "corners" above are really infinitely long straight "edges".