Special Relativity

1) Speed of light is constant in all inertial frames of reference
2) Physical laws must look the same in all inertial frames of reference – there is no experiment that can determine the "absolute" velocity of any inertial frame

$\Rightarrow$ If a flash of light goes off at the origin of some coord system, the outgoing wave-fronts look spherical in all inertial frames.
Equation of wavefront is $r^2 - c^2t^2 = 0$

$\Rightarrow$ (x, y, z, t) coords in one inertial frame K,
(x', y', z', t') coords in another inertial frame K' that moves with velocity $\vec{v} = v\hat{x}$ with respect to K.
What is the transformation that relates coords in K' to coords in K

$y = y'$, $z = z'$

$$\Rightarrow \quad c^2t^2 - x^2 = c^2t'^2 - x'^2$$

$$\Rightarrow \quad \frac{(ct+x)}{(ct'+x')}\frac{(ct-x)}{(ct'-x')} = 1$$

Expect transformation to be linear

$$\Rightarrow \quad ct' + x' = (ct+x) f$$
$$ct' - x' = (ct - x) f^{-1}$$

for some constant f. Write $f = e^{-y}$ y is rapidity

Solve for ct' and x' in terms of ct and x

$$ct' = ct\left(\frac{e^y + e^{-y}}{2}\right) - x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$x' = -ct\left(\frac{e^y - e^{-y}}{2}\right) + x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$ct' = ct\cosh y - x\sinh y$$
$$x' = -ct\sinh y + x\cosh y$$

<u>meaning of parameter y</u>

the origin of K (at $x=0$) has trajectory $x' = -vt'$ in K'

$$\Rightarrow \frac{x'}{t'} = -v$$

from transformation above, with $x=0$, we get

$$\frac{x'}{ct'} = \frac{-ct\sinh y}{ct\cosh y} = -\tanh y$$

so $\frac{v}{c} = \tanh y$

$$\Rightarrow \cosh y = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\sinh y = \left(\frac{v}{c}\right)\gamma$$

<u>Lorentz Transformation</u>
$$\begin{cases} ct' = \gamma ct - \gamma\left(\frac{v}{c}\right)x \\ x' = -\gamma\left(\frac{v}{c}\right)ct + \gamma x \end{cases}$$

Inverse transform obtained by taking $v \to -v$ in above

$$\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right) x' \\ x = \gamma\left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}$$

4 - vectors

4-position: $x_\mu = (x_1, x_2, x_3, ict)$ $\quad x_4 \equiv ict$

$x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2t^2$ Lorentz invariant scalar – has same value in all inertial frames

Lorentz transf is

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 + i\left(\frac{v}{c}\right)x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\left(\frac{v}{c}\right)x_1\right) \end{aligned}\right\}$$

linear transf, can be represented by a matrix

or $\quad x_\mu' = a_{\mu\nu}(L)\, x_\nu$

↑ matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $\quad x_\mu = a_{\mu\nu}(L^{-1})\, x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $\quad a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose

More generally

Since x_μ^2 is Lorentz invariant scalar,

$$x'^2_\mu = a_{\mu\nu}(L)\, a_{\mu\lambda}(L)\, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow \quad a_{\mu\nu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\nu\mu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\mu\nu} = a^{-1}_{\mu\nu}(L)$$ transpose = inverse

$a_{\mu\nu}$ is 4×4 orthogonal matrix

If L_1 is a Lorentz transf from K to K'

L_2 is a Lorentz transf from K' to K''

Then the Lorentz transf from K to K'' is given by the matrix

$$a(L_2 L_1) = a(L_2)\, a(L_1)$$

if $L_1 \equiv L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow \quad a^{-1}(L) = a(L^{-1})$$

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2$$ Lorentz invariant scalar

$$ds^2 = dt^2 \left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2 \right]$$

$$ds^2 = \frac{dt^2}{\gamma^2}$$

$$\boxed{ds = \frac{dt}{\gamma}}$$ proper time interval

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

4-velocity $\quad u_\mu \equiv \dfrac{dx_\mu}{ds} \equiv \mathring{x}_\mu$

$$= \gamma \frac{dx_\mu}{dt}$$

space components $\vec{u} = \gamma \vec{v}$

$$u_4 = ic\gamma$$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2 (v^2 - c^2)$$

$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$

4-acceleration $\quad a_\mu \equiv \dfrac{du_\mu}{ds} = \gamma \dfrac{du_\mu}{dt}$

4-gradient $\quad \dfrac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, \dfrac{-i}{c}\dfrac{\partial}{\partial t}\right)$

proof $\dfrac{\partial}{\partial x_\mu}$ is a 4-vector

$$\frac{\partial}{\partial x_\mu'} = \frac{\partial x_\lambda}{\partial x_\mu'} \frac{\partial}{\partial x_\lambda} \qquad \text{but} \quad \frac{\partial x_\lambda}{\partial x_\mu'} = a_{\lambda\mu}(L^{-1}) = a_{\mu\lambda}(L)$$

$$= a_{\mu\lambda}(L) \frac{\partial}{\partial x_\lambda}$$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu}\right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad \text{wave equation operator!}$$

inner products

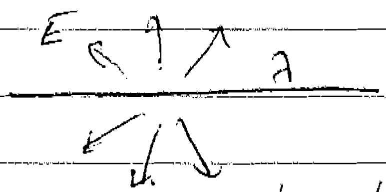
If U_μ and V_μ are 4-vectors, then $U_\mu V_\mu$ is Lorentz invariant scalar

Electromagnetism

Clearly $\vec{E}$ + $\vec{B}$ must transform with each other under Lorentz transf.

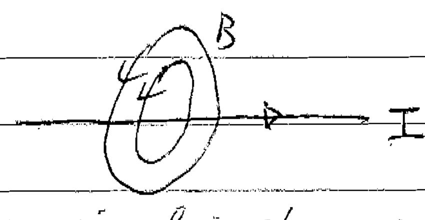
in inertial frame K

stationary line charge λ



cylindrical outward electric field
no B-field

in frame K' moving with $\vec{v}$ ∥ to wire



moving line charge gives current
⇒ B circulating around wire
as well as outward radial E

Lorentz force

$$\vec{F} = q\vec{E} + q\,\frac{\vec{v}}{c}\times\vec{B}$$

What is the velocity $\vec{v}$ here? velocity with respect to what inertial frame? Clearly $\vec{E}$ and $\vec{B}$ must change from one inertial frame to another if this force law can make sense.

Charge density

Consider charge ΔQ contained in a vol ΔV.
ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame

$\Delta Q = \overset{\circ}{\rho}\, \Delta \overset{\circ}{V}$ $\qquad$ $\overset{\circ}{\rho}$ is charge density in the rest frame
$\Delta\overset{\circ}{V}$ is volume in the rest frame

$\overset{\circ}{\rho}$ is Lorentz invariant by definition
Now transform to another frame moving with $\vec{v}$
with respect to rest frame

ΔQ remains the same

$$\Delta V = \frac{\Delta \overset{\circ}{V}}{\gamma} \qquad \text{volume contracts in direction} \parallel \text{to } \vec{v}$$

$$\rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \overset{\circ}{V}}\gamma = \overset{\circ}{\rho}\gamma$$

Current density is $\vec{j} = \rho\vec{v} = \gamma\vec{v}\cdot\frac{\rho}{\gamma} = \overset{\circ}{\rho}\,\vec{u}$

Define <u>4-current</u> $\boxed{j_\mu = (\vec{j}, ic\rho)} = \overset{\circ}{\rho}(\vec{u} \quad ic\gamma)$
$= \overset{\circ}{\rho}\, u_\mu$

It is 4-vector since u_μ is 4-vector and $\overset{\circ}{\rho}$ is Lorentz invariant scalar.

charge conservation

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t} = \boxed{\frac{\partial j_\mu}{\partial x_\mu} = 0}$$

Equation for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{j}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho$$

$\frac{\partial^2}{\partial x_\mu^2} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$ is Lorentz invariant operator

4-potential $\boxed{A_\mu = (\vec{A},\ i\phi)}$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = \boxed{-\frac{4\pi}{c}j_\mu = \frac{\partial^2 A_\mu}{\partial x_{\cancel{\lambda}}^2}}$$

Lorentz gauge condition is

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{c\,\partial t} = \frac{\partial A_\mu}{\partial x_\mu} = 0$$

Electric and magnetic fields

$$B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \qquad i,j,k \text{ cyclic permutation of } 1,2,3$$

$$E_i = -\frac{\partial\phi}{\partial x_i} - \frac{\partial A_i}{c\,\partial t} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)$$

Define field stress tensor

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}}$$

$$= \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4x4 anti symmetric 2nd rank tensor

Inhomogeneous Maxwell's equations can be written in the form

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} j_\mu} \Rightarrow \left[\begin{array}{l} \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \end{array}\right]$$

$$= \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \underbrace{\frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right)}_{=0} - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

$$\Rightarrow -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = \frac{4\pi}{c} j_\mu$$ agrees with previous equation for A_μ

transformation law for 2nd rank tensor $F_{\mu\nu}$

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$

use $A'_\mu = a_{\mu\sigma} A_\sigma$

$$\frac{\partial}{\partial x'_\mu} = a_{\mu\lambda}\frac{\partial}{\partial x_\lambda}$$

$$= a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma} - a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}$$ ← lets one find $\vec{E}'$ and $\vec{B}'$ if one knows $\vec{E}$ and $\vec{B}$

For nth rank tensor

$$T'_{\mu_1\mu_2\cdots\mu_n} = a_{\mu_1\nu_1} a_{\mu_2\nu_2} \cdots a_{\mu_n\nu_n} T_{\nu_1\nu_2\cdots\nu_n}$$