Magnetostatics

$$\vec{B} = \vec{\nabla}\times\vec{A}\ , \quad \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} \end{cases}$$ Ampere's law (statics only!)

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j}$$

can write $\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

where by $\nabla^2\vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2\vec{A}$ only has a simple expression in Cartesian coords

Aside

If tried to write it in spherical coords, for example, one has

$$\nabla^2\vec{A} = \nabla^2(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi})$$
$$= \cancel{(\nabla^2 A_r)\hat{r} + A_r(\nabla^2\hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2\hat{\theta})}$$
$$\cancel{+ (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2\hat{\varphi})}$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

one could compute $\nabla^2\hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla}\cdot\vec{A} = 0$, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \boxed{-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}}$$ Poisson's equation!

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of eqn makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ three equations for A_x, A_y, A_z !

for localized current sources $\vec{j}(r) \to 0$ as $r \to \infty$

<u>Multipole expansion</u> - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use <u>vector spherical harmonics</u> - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{1/2}} = \frac{1}{r}\,\frac{1}{\left[1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1^{st} order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \overset{①}{\int \frac{d^3r'}{c} \frac{\vec{J}(\vec{r}\,')}{r}} + \overset{②}{\int \frac{d^3r'}{c} \vec{J}(\vec{r}\,') \frac{(\vec{r}\cdot\vec{r}\,')}{r^3}} + \cdots$$

Consider term ①

$$\int d^3r\, \vec{J}(\vec{r}) \qquad \int d^3r (\vec{J}\cdot\vec{\nabla})\vec{r} \qquad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

$$\parallel$$

write $\int d^3r\, J_i(r) = \sum_{j=1}^{3} \int d^3r\, J_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

$$= \sum_j \left\{ \oint_S da\, J_j r_i - \int d^3r \frac{\partial J_j}{\partial r_j} r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{J}$ sufficiently localized ie $\vec{J}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{J} = 0$

So $\int d^3r\, \vec{J}(\vec{r}) = 0$ in magnetostatics
monopole term vanishes

term (2)

$$\int d^3r \ \vec{j}(\vec{r})\,\vec{r} \qquad \text{tensor}$$

Consider $\displaystyle\int d^3r\ j_i r_j = \sum_k \int d^3r\ j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

$$= \sum_k \left\{ \oint_S da\ j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k}(j_k r_j)\, r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized

$$= -\sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla}\cdot\vec{j} = 0$ in magnetostatics $\qquad = \delta_{jk}$

$$= -\int d^3r\ j_j r_i$$

So $\displaystyle\int d^3r\ j_i r_j = -\int d^3r\ j_j r_i$

$$= \frac{1}{2}\int d^3r\,(j_i r_j - j_j r_i)$$

So

$$\int d^3r'\ j_i(\vec{r}')(\vec{r}\cdot\vec{r}') = \sum_j r_j \int d^3r'\ j_i(\vec{r}')\, r_j'$$

$$= \sum_j \frac{1}{2}\int d^3r'\,(j_i r_j r_j' - r_j j_j r_i')$$

$$= \frac{1}{2}\int d^3r' \left(j_i (\vec{r}\cdot\vec{r}') - r_i' (\vec{r}\cdot\vec{j}) \right)$$

use triple product rule

$$\vec{r}\times(\vec{r}'\times\vec{J}) = \vec{r}'(\vec{r}\cdot\vec{J}) - \vec{J}(\vec{r}\cdot\vec{r}')$$

to rewrite as

$$\int d^3r'\,\vec{J}\,(\vec{r}\cdot\vec{r}') = -\frac{1}{2}\,\vec{r}\times\left[\int d^3r'\,\vec{r}'\times\vec{J}(\vec{r}')\right]$$

define the magnetic dipole moment as

$$\boxed{\vec{m} = \frac{1}{2c}\int d^3r'\,\vec{r}'\times\vec{J}(\vec{r}')}$$

In magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}(\vec{r}) = -\frac{\vec{r}\times\vec{m}}{r^3} = \frac{\vec{m}\times\vec{r}}{r^3} = \frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(\vec{m}\times\frac{\vec{r}}{r^3}\right)$$

$$B_i = \varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,m_l\,\frac{r_m}{r^3}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\,m_l\,\frac{r_m}{r^3}$$

$$= m_i\,\partial_j\left(\frac{r_j}{r^3}\right) - m_j\,\partial_j\left(\frac{r_i}{r^3}\right)$$

$$= m_i\left[-4\pi\delta(\vec{r})\right] - m_j\left[\frac{\delta_{ij}}{r^3} - \frac{3r_i}{r^4}\,\partial_j r\right]$$

$$= \underset{\text{far from source}}{0} \quad -\frac{m_i}{r^3} + 3\frac{r_i}{r^4}\frac{r_j}{r}m_j$$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{any two indices equal} \\ +1, & ijk \text{ even permutation of } 123 \\ -1, & ijk \text{ odd permutation } 123 \end{cases}$$

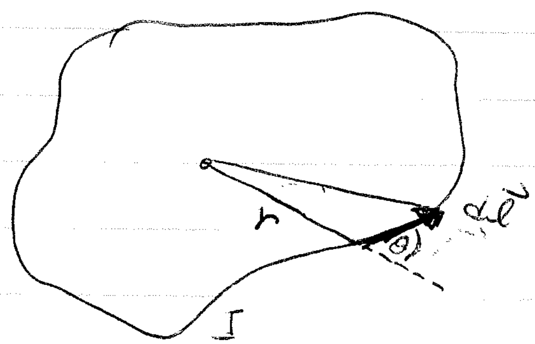
$$(\vec{A}\times\vec{B})_i = \varepsilon_{ijk}A_jB_k$$

$$\boxed{\vec{B} = \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}}$$

same form as $\vec{E}$ from electric dipole $\vec{p}$

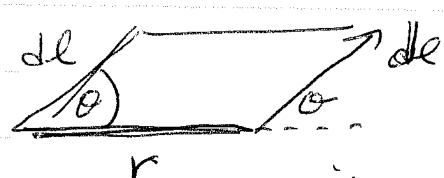
For a <u>current loop in a plane</u> (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r}\times\vec{j} = \frac{1}{2c} I \oint \vec{r}\times d\vec{l}$$



area of triangle is $\frac{1}{2} r\, dl \sin\theta$

$= \frac{1}{2} |\vec{r}\times d\vec{l}|$



area of ~~rectangle~~ trapazoid is $r\, dl \sin\theta$

$\Rightarrow \quad \vec{m} = \frac{1}{c} I (\text{area})\, \hat{m}$

area of loop

outward normal (direction given by right hand rule with respect to direction of current) magnetic dipole moment $\vec{m}$ is independent of location of origin.

$$\vec{r}\,' = \vec{r} + \vec{d} \qquad \text{new coord}$$

$$\vec{m}\,' = \frac{1}{2c}\int d^3r' \,(\vec{r}\,' \times \vec{j}) = \frac{1}{2c}\int d^3r \,(\vec{r} + \vec{d}) \times \vec{j}$$

$$= \frac{1}{2c}\int d^3r \,\vec{r} \times \vec{j} + \frac{1}{2c}\vec{d} \times \left[\int d^3r \,\vec{j}\,\right]$$

$$\vec{m}\,' = \vec{m} + 0 \quad \text{as} \quad \int d^3r \,\vec{j} = 0$$

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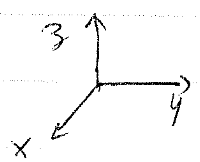
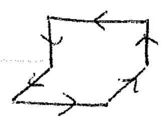
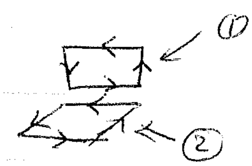
for planar loop $\quad \vec{m} = \frac{I}{c} a \,\hat{n} \quad$ where a = area
$\hat{n}$ = outward normal

Can also apply to get $\vec{m}$ for <u>piecewise planar loops</u>



 = superposition of 

$$\vec{m} = \vec{m}_1 + \vec{m}_2 \qquad \vec{m}_1 = \frac{I}{c} a_1 \hat{x}$$
$$\vec{m}_2 = \frac{I}{c} a_2 \hat{z}$$

$$\Rightarrow \quad \vec{m} = \frac{I}{c}(a_1 \hat{x} + a_2 \hat{z})$$