So

$$② = \sum_{i\in n} q_i \vec{v}_{ni}\, f(\vec{r}-\vec{r}_n) = \frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle + (\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(r-r_n)\rangle$$

2^{nd} term is $\sum_\alpha v_{n\alpha} \frac{\partial}{\partial r_\alpha}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$

$$③ = -\vec{v}_n \Big(\sum_{i\in n} q_i \vec{r}_{ni}\Big)\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n) = -\vec{v}_n \left(\vec{p}_n\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n)\right)$$

$$= -\vec{v}_n\, \vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle = \sum_\alpha \vec{v}_n \frac{\partial}{\partial r_\alpha}\langle p_{n\alpha}\delta(\vec{r}-\vec{r}_n)\rangle$$

$$④ = -\vec{\nabla} f(\vec{r}-\vec{r}_n)\cdot \sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni}$$

We have seen the tensor $\sum_i q_i \vec{r}_{ni}\vec{v}_{ni}$ before when we considered the magnetic dipole moment

$$\sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni} = \int d^3r\, \vec{r}\,\vec{j}$$

where $\vec{j}(\vec{r}) \equiv \sum_{i\in n} q_i \vec{v}_{ni}\,\delta(\vec{r}-\vec{r}_{ni})$ is current density with respect to center of mass of molecule

We had $\int d^3r\, \vec{r}\,\vec{j} = -\int d^3r\, \vec{j}\,\vec{r} - \int d^3r\, (\vec{\nabla}\cdot\vec{j})\, \vec{r}\,\vec{r}$

in statics, $\vec{\nabla}\cdot\vec{j} = 0$

in general $\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t}$

$$\int d^3r\, \vec{r}\,\vec{j} = -\int d^3r\, \vec{j}\,\vec{r} + \int d^3r\, \frac{\partial\rho}{\partial t}\,\vec{r}\,\vec{r}$$

$$= -\int d^3r\, \vec{j}\,\vec{r} + \frac{\partial}{\partial t}\left[\int d^3r\, \rho\, \vec{r}\,\vec{r}\right]$$

although this is not zero, it is a <u>quadrapole</u> term of the same order as the terms we dropped when we truncated

$\sim O\left(\frac{a_0}{L}\right)^2$

So $\int d^3r\, \vec{r}\,\vec{J} \simeq -\int d^3r\, \vec{J}\,\vec{r}$ ignoring the quadrapole term

$$= \frac{1}{2}\int d^3r \left[\vec{r}\,\vec{J} - \vec{J}\,\vec{r} \right]$$

$$\sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = \frac{1}{2}\sum_{i\in n} q_i \left[\vec{r}_{ni}\,\vec{v}_{ni} - \vec{v}_{ni}\,\vec{r}_{ni} \right]$$

$$-\vec{\nabla} f(r-r_n) \cdot \sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = -\vec{\nabla} f(\vec{r}-\vec{r}_n) \cdot \frac{1}{2}\sum_i q_i \left[\vec{r}_{ni}\,\vec{v}_{ni} - \vec{v}_{ni}\,\vec{r}_{ni} \right]$$

$$= \frac{-1}{2}\sum_{i\in n} q_i \left[(\vec{\nabla} f \cdot \vec{r}_{ni})\,\vec{v}_{ni} - (\vec{\nabla} f \cdot \vec{v}_{ni})\,\vec{r}_{ni} \right]$$

$$= -\frac{1}{2}\sum_{i\in n} q_i \; \vec{\nabla} f \times (\vec{v}_{ni} \times \vec{r}_{ni})$$ triple product rule

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n) \times \frac{1}{2}\sum_{i\in n} \vec{r}_{ni} \times \vec{v}_{ni}\, q_i$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n) \times \frac{1}{2}\int d^3r\, \vec{r} \times \vec{J}$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n) \times c\,\vec{m}_n$$ where $\vec{m}_n = \frac{1}{2c}\sum_{i\in n} \vec{r}_{ni} \times \vec{v}_{ni}\, q_i$

is magnetic dipole moment of molecule n

$$= \vec{\nabla} \times f(\vec{r}-\vec{r}_n)\, c\,\vec{m}_n$$

$$= \vec{\nabla} \times \left\langle c\,\vec{m}_n\, \delta(\vec{r}-\vec{r}_n) \right\rangle$$

Adding all the pieces

$$\langle \vec{j}_n \rangle = \underset{①}{\langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n)\rangle} + \underset{④}{c\vec{\nabla}\times\langle \vec{m}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

$$+ \underset{②}{\frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle} + \underset{②}{(\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

$$- \underset{③}{\vec{v}_n\, \vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

Define $\vec{\mathbb{M}}(\vec{r}) \equiv \sum_n \langle \vec{m}_n \delta(\vec{r}-\vec{r}_n)\rangle$ average magnetization density

$\vec{\mathbb{P}}(\vec{r}) \equiv \sum_n \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$ polarization density, as before

$$\sum_n \langle \vec{j}_n \rangle = \sum_n \langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n)\rangle + c\vec{\nabla}\times\vec{\mathbb{M}} + \frac{\partial\vec{\mathbb{P}}}{\partial t}$$

$$+ \sum_n \left[(\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle - \vec{v}_n\, \vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle\right]$$

See Jackson (6.96) for additional electric quadrupole terms

The last term on the right hand side is usually small and ignored. This is because the molecular velocities $\vec{v}_n$ are usually small, and randomly oriented, so that they average to zero. (see Jackson (6.100) for case of net translation of dielectric, $\vec{v}_n$ = const all n)

Define macroscopic current density

$$\vec{J}(\vec{r},t) = \Big\langle \sum_{i \in \text{free}} q_i \vec{v}_i \,\delta(\vec{r}-\vec{r}_i) \Big\rangle + \Big\langle \sum_{\substack{n \\ \text{molecs}}} q_n \vec{v}_n \,\delta(\vec{r}-\vec{r}_n) \Big\rangle$$

current of free charges

current of molecular drifting if molecules are charged

Then $\langle \vec{j}_0 \rangle = \vec{J} + c\,\vec{\nabla}\times\vec{M} + \dfrac{\partial \vec{P}}{\partial t}$

Ampere's law becomes upon averaging

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle \vec{j}_0 \rangle + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$= \frac{4\pi}{c}\vec{J} + 4\pi\vec{\nabla}\times\vec{M} + \frac{4\pi}{c}\frac{\partial \vec{P}}{\partial t} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\left(\vec{B} - 4\pi\vec{M}\right) = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E} + 4\pi\vec{P}\right)$$

define $\boxed{\vec{H} \equiv \vec{B} - 4\pi\vec{M}}$ to get

$$\boxed{\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}}$$

$\boxed{\vec{D} = \vec{E} + 4\pi\vec{P}}$ as before

official nomenclature: $\vec{B}$ is the magnetic induction

$\vec{H}$ is the magnetic field

common usage: both $\vec{H}$ and $\vec{B}$ are called magnetic field

When atoms have intrinsic magnetic moments due to electron spin, we can add these to $\vec{M}$ in obvious way

When molecules are neutral, $q_n = 0$, the "bound current" is given by

$$\vec{J}_{bound} = \sum_n \langle \vec{j}_n \rangle = c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of <u>bound charge</u>

$$\vec{\nabla}\cdot\vec{J}_{bound} = c\,\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) + \vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t}$$

$$= 0 + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P})$$

$$= -\frac{\partial \rho_{bound}}{\partial t}$$ where $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$ is bound charge density

So $$\boxed{\vec{\nabla}\cdot\vec{J}_{bound} + \frac{\partial \rho_{bound}}{\partial t} = 0}$$

and bound charge is conserved.

Since total average charge must be conserved, ie

$$\vec{\nabla}\cdot\langle \vec{j}_0\rangle - \frac{\partial \langle \rho_0\rangle}{\partial t} = 0\,,$$ and $\langle \vec{j}_0\rangle = \vec{J} + \vec{J}_{bound}$

free current

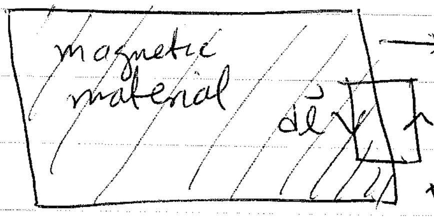
$\langle \rho_0\rangle = \rho + \rho_{bound}$

free charge

$$\Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

free charge is also conserved

At a surface of a magnetic material



$\rightarrow \hat{n}$ outward normal to surface

take $\hat{z} \equiv d\hat{l} \times \hat{n}$ out of page

Amperean loop C bounding surface S of area da

$$c\int_S da\, \hat{z}\cdot(\vec{\nabla}\times\vec{M}) = \int_S da\, \hat{z}\cdot\vec{J}_{bound} = da\, \hat{z}\cdot\vec{J}_{bound}$$

$$= (d\vec{l}\times\hat{n})\cdot\vec{K}_{bound}$$ as width of loop $\rightarrow 0$

$$= (\hat{n}\times\vec{K}_{bound})\cdot d\vec{l}$$

But by Stokes theorem

$$c\int_S da\, \hat{z}\cdot(\vec{\nabla}\times\vec{M}) = c\int_C d\vec{l}\cdot\vec{M} = c\, d\vec{l}\cdot\vec{M}$$ since width $\rightarrow 0$ and $\vec{M}=0$ outside

$$\Rightarrow c\, d\vec{l}\cdot\vec{M} = (\hat{n}\times\vec{K}_{bound})\cdot d\vec{l}$$ for any $d\vec{l}$ in plane of surface

$$\Rightarrow c\vec{M}_t = \hat{n}\times\vec{K}_{bound}$$

where $\vec{M}_t$ is component of $\vec{M}$ tangential to the surface (since $\vec{K}_b$ is in plane of surface, $\hat{n}\times\vec{K}$ is also entirely in the plane of the surface)

$$\Rightarrow c\,\hat{n}\times\vec{M}_t = c\,\hat{n}\times\vec{M} = \hat{n}\times(\hat{n}\times\vec{K}_{bound})$$
$$= -\vec{K}_{bound}$$

$$\Rightarrow \boxed{\begin{aligned}\vec{K}_{bound} &= c\,\vec{M}\times\hat{n}\\ \vec{J}_{bound} &= c\,\vec{\nabla}\times\vec{M}\end{aligned}}$$

Total bound charge vanishes

$$Q_{bound} = \int_V d^3r\, \rho_{bound} + \int_S da\, \sigma_{bound}$$

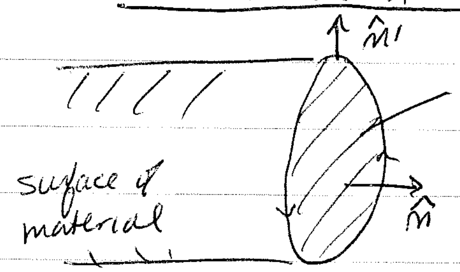
↑ vol of dielectric ← surface of dielectric

$$= \int_V d^3r\, -\vec{\nabla}\cdot\vec{P} + \int da\, \hat{n}\cdot\vec{P}$$

but by Gauss theorem $\int_V d^3r\, \vec{\nabla}\cdot\vec{P} = \int da\, \hat{n}\cdot\vec{P}$

so $Q_{bound} = -\int da\, \hat{n}\cdot\vec{P} + \int da\, \hat{n}\cdot\vec{P} = 0$

Total bound current vanishes



crossectional area of magnetic material

S - bounding curve is C

$\hat{n}$ is normal to crossection
$\hat{n}'$ is normal to surface

total current flowing through S is

$$\int_S da\, \hat{n}\cdot\vec{j}_{bound} + \int_C dl\, \vec{K}_{bound}\cdot\hat{n}$$

$$= c\int_S da\, \hat{n}\cdot(\vec{\nabla}\times\vec{M}) + c\int_C dl\, \hat{n}\cdot(\vec{M}\times\hat{n}')$$

$$= c\int_C d\vec{l}\cdot\vec{M} + c\int_C dl\, \underbrace{(\hat{n}'\times\hat{n})}_{=-\hat{t}}\cdot\vec{M}$$

$= -\hat{t}$ unit tangent, $d\vec{l} = dl\,\hat{t}$

$$= c\int_C d\vec{l}\cdot\vec{M} - c\int_C d\vec{l}\cdot\vec{M} = 0$$

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad\qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \qquad\qquad \vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

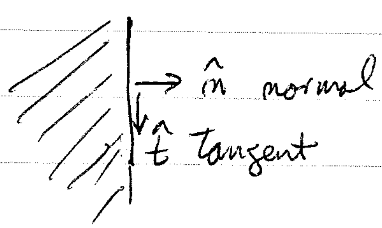
where ρ and $\vec{j}$ are macroscopic charge + current densities
do not include bound charges or currents

$\vec{D} = \vec{E} + 4\pi\vec{P}$, $\vec{P}$ is polarization density

$\vec{H} = \vec{B} - 4\pi\vec{M}$, $\vec{M}$ is magnetization density

Boundary conditions for statics

electrostatics: at surface of a dielectric, or at interface between two different dielectrics



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \hat{t}\cdot\vec{E}_{above} = \hat{t}\cdot\vec{E}_{below}$

tangential component $\vec{E}$ is continuous

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \Rightarrow \hat{n}\cdot(\vec{D}_{above} - \vec{D}_{below}) = 4\pi\sigma$

normal component of $\vec{D}$ jumps by $4\pi\sigma$

magnetostatics: at surface or interface of magnetic materials

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \hat{n}\cdot\vec{B}_{above} - \hat{n}\cdot\vec{B}_{below}$

normal component of $\vec{B}$ is continuous

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} \Rightarrow \hat{t}\cdot(\vec{H}_{above} - \vec{H}_{below}) = \frac{4\pi}{c}(\vec{K}\times\hat{n})\cdot\hat{t}$

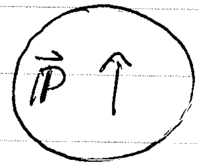
tangential component of $\vec{H}$ jumps by $\frac{4\pi}{c}\vec{K}\times\hat{n}$

if $\sigma = 0$, ie no free surface charge, then $\hat{n}\cdot\vec{D}$ continuous

if $\vec{K} = 0$, ie no free surface current, then $\hat{t}\cdot\vec{H}$ continuous

Examples

① Uniformly polarized sphere of radius R $\qquad \vec{\mathbb{P}} = \mathbb{P}\hat{z}$



bound charge $\quad \rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}} = 0 \quad$ as $\vec{\mathbb{P}}$ constant

$$\sigma_b = \hat{n}\cdot\vec{\mathbb{P}} = \hat{r}\cdot\vec{\mathbb{P}} = \mathbb{P}\cos\theta$$

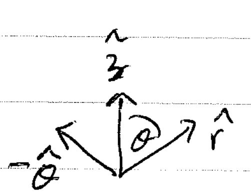
we saw earlier that a sphere with surface charge $\sigma(\theta) = \sigma_0\cos\theta$ gives an electric field like a pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 \mathbb{P}\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ -\dfrac{4\pi\mathbb{P}}{3}\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{p} = \frac{4}{3}\pi R^3\vec{\mathbb{P}}$

check behavior at boundary

Tangential component $\vec{E}$



$$\vec{E}^{\,t}_{above} = \left(\frac{4}{3}\pi R^3\mathbb{P}\right)\frac{\sin\theta\,\hat{\theta}}{R^3} = \frac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$

$$\vec{E}^{\,t}_{below} = -\frac{4\pi\mathbb{P}}{3}(\hat{z}\cdot\hat{\theta})\,\hat{\theta} = \frac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$

$\Rightarrow$ Tangential component $\vec{E}$ is continuous

normal component of $\vec{D}$

outside: $\vec{\mathbb{P}} = 0 \Rightarrow \vec{D} = \vec{E}$

$$\Rightarrow \hat{n}\cdot\vec{D} = \hat{r}\cdot\vec{E} = \left(\frac{4}{3}\pi R^3\mathbb{P}\right)\frac{2\cos\theta\,\hat{r}}{R^3} = \frac{8}{3}\pi\,\mathbb{P}\cos\theta,$$

inside: $\vec{E} = -\frac{4\pi\vec{P}}{3} \Rightarrow \vec{P} = -\frac{3}{4\pi}\vec{E}$

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \vec{E} - 3\vec{E} = -2\vec{E} = \frac{8\pi}{3}P\,\hat{z}$$

$$\hat{n}\cdot\vec{D} = \hat{r}\cdot\left(\frac{8\pi}{3}P\hat{z}\right) = \frac{8\pi}{3}P\cos\theta$$

$\Rightarrow$ normal component $\vec{D}$ is continuous

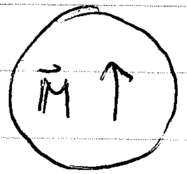
Note: normal component of $\vec{E}$ should jump by $4\pi\sigma_b = 4\pi P\cos\theta$

to check this: $\hat{n}\cdot\vec{E}_{\text{inside}} = \hat{r}\cdot\left(-\frac{4}{3}\pi P\hat{z}\right) = -\frac{4}{3}\pi P\cos\theta$

$$\hat{n}\cdot\left(\vec{E}^{\,\text{above}} - \vec{E}^{\,\text{below}}\right) = \frac{8}{3}\pi P\cos\theta + \frac{4}{3}\pi P\cos\theta$$

$$= \frac{12}{3}\pi P\cos\theta = 4\pi P\cos\theta \overset{4\pi}{=} \sigma_b(\theta)$$

② Uniformly magnetized sphere of radius R $\quad \vec{M} = M\hat{z}$



bound current $\quad \vec{J}_b = c\,\vec{\nabla}\times\vec{M} = 0$ as $\vec{M}$ constant

$$\vec{K}_b = c\,\vec{M}\times\hat{n} = cM(\hat{z}\times\hat{r})$$
$$= cM\sin\theta\,\hat{\phi}$$

we saw earlier that a sphere with surface current $\vec{K}_b = K_0\sin\theta\,\hat{\theta}$ gives a magnetic field that is pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{B}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 M\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ \frac{8}{3}\pi M\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$

Check behavior at boundary

normal component of $\vec{B}$

$$\hat{n}\cdot\vec{B}^{\,above} = \hat{r}\cdot\vec{B}^{\,above} = \frac{8}{3}\pi M\cos\theta$$

$$\hat{n}\cdot\vec{B}^{\,below} = \hat{r}\cdot\vec{B}^{\,below} = \frac{8}{3}\pi M(\hat{r}\cdot\hat{z}) = \frac{8}{3}\pi M\cos\theta$$

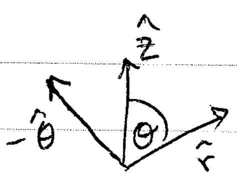
$\Rightarrow$ normal component of $\vec{B}$ is continuous

tangential component of $\vec{H}$

outside: $\vec{M}=0 \Rightarrow \vec{H}=\vec{B}$

$$\vec{H}^{\,t}_{above} = \left(\frac{4}{3}\pi M\right)\sin\theta\,\hat{\theta}$$

inside: $$\vec{H} = \vec{B} - 4\pi\vec{M} = \vec{B} - 4\pi\left(\frac{3}{8\pi}\vec{B}\right) = \vec{B} - \frac{3}{2}\vec{B} = -\frac{1}{2}\vec{B}$$
$$= -\frac{4\pi}{3}M\hat{z}$$



so $$\vec{H}^{\,t}_{below} = -\frac{4\pi}{3}M(\hat{z}\cdot\hat{\theta}) = \frac{4\pi}{3}M\sin\theta\,\hat{\theta}$$

$\Rightarrow$ tangential component $\vec{H}$ is continuous

Note: tangential component $\vec{B}$ should jump by $\frac{4\pi}{c}\vec{K}_b\times\hat{n} = 4\pi M\sin\theta\,\hat{\theta}$

inside:

to check: $$\vec{B}^{\,t}_{below} = \frac{8}{3}\pi M(\hat{z}\cdot\hat{\theta})\hat{\theta} = -\frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$\vec{H}^{\,t}_{above} = \vec{B}^{\,t}_{above} \Rightarrow$ $$\vec{B}^{\,t}_{above} - \vec{B}^{\,t}_{below} = \frac{4}{3}\pi M\sin\theta\,\hat{\theta} + \frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$$= 4\pi M\sin\theta\,\hat{\theta} = \frac{4\pi}{c}\vec{K}_b$$