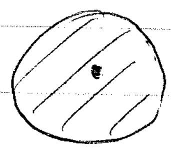
point charge within a dielectric sphere



pt charge q at center of dielectric sphere of radius R, dielectric const ϵ

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad = \quad \oint_S da\,\hat{n}\cdot\vec{D} = 4\pi Q_{encl}$$

From symmetry $\vec{D}(\vec{r}) = D(r)\,\hat{r}$

$$\oint_S da\,\hat{n}\cdot\vec{D} = 4\pi r^2 D(r) = 4\pi q$$

sphere of radius r

$$\vec{D} = \frac{q}{r^2}\hat{r} \qquad \text{all } r$$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{\epsilon r^2}\hat{r} & r<R \\ \dfrac{q}{r^2}\hat{r} & r>R \end{cases}$$

can check that tangential component of $\vec{E}$ is continuous and normal component of $\vec{D}$ is continuous as there is no free σ at surface of dielectric.

normal component of $\vec{E}$ jumps by

$$\hat{n}\cdot(\vec{E}^{above}-\vec{E}^{below}) = \frac{q}{R^2} - \frac{q}{\epsilon R^2} = \frac{q}{R^2}\left(1-\frac{1}{\epsilon}\right) = \frac{q}{R^2}\left(\frac{\epsilon-1}{\epsilon}\right)$$

$$= \frac{q}{R^2}\left(\frac{4\pi\chi_e}{1+4\pi\chi_e}\right) = 4\pi\sigma_{total} = 4\pi\sigma_b$$

$$\Rightarrow \sigma_b = \frac{q}{4\pi R^2}\left(\frac{4\pi\chi_e}{1+4\pi\chi_e}\right) = \frac{q\,\chi_e}{R^2\epsilon}$$

We can check this directly

$$\vec{P} = \chi_e \vec{E} = \frac{\chi_e}{\epsilon} \frac{q}{r^2} \hat{r}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{\chi_e}{\epsilon} q \, 4\pi \delta(\vec{r})$$

bound charge at origin $q_b = -\frac{\chi_e}{\epsilon} 4\pi q$

total charge at origin is $q + q_b = q\left(1 - \frac{4\pi \chi_e}{\epsilon}\right)$

$\epsilon = 1 + 4\pi\chi_e$ $\qquad = q\left(\frac{\epsilon - 4\pi\chi_e}{\epsilon}\right) = \frac{q}{\epsilon}$ <u>screened charge</u>

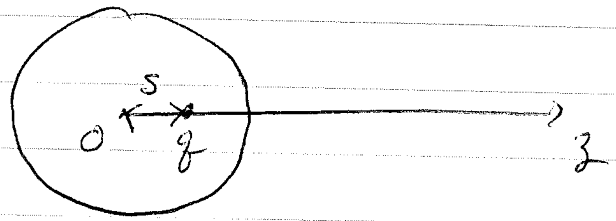
at surface,

$$\sigma_b = \hat{n} \cdot \vec{P} = \frac{\chi_e}{\epsilon} \frac{q}{R^2}$$

agrees with what we get from jump in $\hat{n} \cdot \vec{E}$.

<u>Note:</u> inside the dielectric the $\vec{E}$ field is that of the screened point charge $\frac{q}{\epsilon}$
outside the dielectric $\vec{E}$ is just that of the free charge q - there is no evidence in $\vec{E}_{out}$ that the dielectric even exists!

Now consider same problem but q is off center



what is $\vec{E}$ inside + outside

inside $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ where $\rho = q\,\delta(\vec{r} - s\hat{z})$

$$\vec{D} = \epsilon\vec{E} \Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho/\epsilon$$
$$\vec{E} = -\vec{\nabla}\phi \Rightarrow \nabla^2\phi = -\frac{4\pi\rho}{\epsilon} = -\frac{4\pi q}{\epsilon}\,\delta(\vec{r} - s\hat{z})$$

solution for ϕ will be of the form

$$\phi(\vec{r}) = \frac{q}{\epsilon|\vec{r} - s\hat{z}|} + F(\vec{r})$$

where 1^{st} term is due to the point charge q/ϵ and 2^{nd} term satisfies $\nabla^2 F = 0$ and will be chosen to get the correct behavior at the boundary of the dielectric

Since there is azimuthal symmetry about $\hat{z}$ we can write

$$F(\vec{r}) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

there are no $\frac{b_l}{r^{l+1}}$ terms since F should not diverge at the origin

So inside, $r < R$

$$\phi^{in}(\vec{r}) = \frac{q}{\epsilon|\vec{r} - s\hat{z}|} + \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta)$$

From our discussion of electric multipole expansion, we know we can write for $r > s$,

$$\frac{1}{|\vec{r} - s\hat{z}|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{s}{r}\right)^\ell P_\ell(\cos\theta)$$

So for $\underline{\underline{r > s}}$ (not true for $r < s$!)

$$\phi^{in}(\vec{r}) = \sum_{\ell=0}^{\infty} \left(\frac{q}{\epsilon r}\left(\frac{s}{r}\right)^\ell + a_\ell r^\ell \right) P_\ell(\cos\theta)$$

Outside the sphere there is no charge, so $\vec{\nabla} \cdot \vec{E} = 0$ or $\nabla^2 \phi = 0$

$$\Rightarrow \phi^{out}(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

there are no $a_\ell r^\ell$ terms since $\phi^{out} \to 0$ as $r \to \infty$

To determine the unknown a_ℓ and b_ℓ we use the boundary conditions at surface of dielectric at $r = R$

① <u>tangential component $\vec{E}$</u> is continuous

$$\vec{E} = -\frac{\partial \phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta} = E_r \hat{r} + E_\theta \hat{\theta}$$

$\Rightarrow$ E_θ is continuous at $r = R$

condition that E_θ is continuous is the same condition that ϕ is continuous (check this out for yourself if you are not sure)

as $\vec{E}^{above} - \vec{E}^{below} = 4\pi\sigma_{tot}$

$$\Rightarrow \phi^{in}(R,\theta) = \phi^{out}(R,\theta)$$

$$\frac{q}{\epsilon R}\left(\frac{s}{R}\right)^l + a_l R^l = \frac{b_l}{R^{l+1}}$$

$$\Rightarrow \boxed{b_l = \frac{q}{\epsilon} s^l + a_l R^{2l+1}}$$

<u>normal component</u> $\vec{D}$ is continuous (since free surface charge $\sigma = 0$)

$$\vec{D} = \epsilon \vec{E}$$

$$\Rightarrow \epsilon E_r^{in} = E_r^{out}$$

$$-\epsilon \left.\frac{\partial \phi^{in}}{\partial r}\right|_R = -\left.\frac{\partial \phi^{out}}{\partial r}\right|_R$$

$$\Rightarrow \frac{(l+1) q}{R^2}\left(\frac{s}{R}\right)^l - l \epsilon a_l R^{l-1} = \frac{(l+1) b_l}{R^{l+2}}$$

$$q s^{\ell} - \frac{\ell}{\ell+1} \epsilon a_{\ell} R^{2\ell+1} = b_{\ell}$$

substitute in b_{ℓ} from previous boundary condition

$$q s^{\ell} - \frac{\ell}{\ell+1} \epsilon a_{\ell} R^{2\ell+1} = \frac{q}{\epsilon} s^{\ell} + a_{\ell} R^{2\ell+1}$$

$$q s^{\ell}\left[1 - \frac{1}{\epsilon}\right] = a_{\ell} R^{2\ell+1}\left[1 + \frac{\ell}{\ell+1}\epsilon\right]$$

$$\boxed{a_{\ell} = \frac{q s^{\ell}}{R^{2\ell+1}} \frac{\left[1 - \frac{1}{\epsilon}\right]}{\left[1 + \left(\frac{\ell}{\ell+1}\right)\epsilon\right]}}$$

$$b_{\ell} = \frac{q}{\epsilon} s^{\ell} + a_{\ell} R^{2\ell+1}$$

$$= \frac{q}{\epsilon} s^{\ell} + q s^{\ell} \frac{\left[1 - \frac{1}{\epsilon}\right]}{\left[1 + \left(\frac{\ell}{\ell+1}\right)\epsilon\right]}$$

$$b_{\ell} = \frac{q s^{\ell}}{\epsilon}\left\{1 + \frac{\epsilon - 1}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]$$

$$= \frac{q s^{\ell}}{\epsilon}\left[\frac{\epsilon\left(1 + \frac{\ell}{\ell+1}\right)}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]$$

$$\boxed{b_{\ell} = q s^{\ell}\left[\frac{1 + \left(\frac{\ell}{\ell+1}\right)}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]}$$

Check the result:

as $s \to 0$, should recover previous answer

for $s = 0$, $a_\ell = b_\ell = 0$ for all $\ell \neq 0$

$$a_0 = \frac{q}{R}\left[1 - \frac{1}{\epsilon}\right]$$

$$b_0 = q$$

so $$\phi^{in}(\vec{r}) = \frac{q}{\epsilon r} + \frac{q}{R}\left[1 - \frac{1}{\epsilon}\right]$$

$$\vec{E}^{in} = -\vec{\nabla}\phi^{in} = \frac{q}{\epsilon r^2}\hat{r} \quad \text{as before}$$

$$\phi^{out}(\vec{r}) = \frac{q}{r}$$

$$\vec{E}^{out} = -\vec{\nabla}\phi^{out} = \frac{q}{r^2}\hat{r} \quad \text{as before}$$

Note: the constant that is the 2nd term in ϕ^{in} is just what is needed to make ϕ continuous at $r = R$

another check:

let $\epsilon \to \infty$ this models a conductor!

again one finds $a_\ell = b_\ell = 0$ for all $\ell \neq 0$

$$a_0 = \frac{q}{R}$$

$$b_0 = q$$

$$\phi^{in}(\vec{r}) = \sum_\ell \frac{q}{\epsilon r}\left(\frac{s}{r}\right)^\ell P_\ell + \frac{q}{R} \to \frac{q}{R} \quad \text{as } \epsilon \to \infty$$

$$\Rightarrow E^{in}(\vec{r}) = 0 \quad \text{as } \phi^{in} \text{ is a constant.}$$

$$\phi^{out}(\vec{r}) = \frac{q}{r} \Rightarrow \vec{E}^{out} = \frac{q}{r^2}\hat{r}$$

field outside is like point charge q at the origin, independent of where q is inside the sphere. This is the correct behavior of a conductor. The mobile charges in the conductor completely screen the q inside, and leave a uniform surface charge $\sigma_b = \frac{q}{4\pi R^2}$ on the surface.

Magnetostatics

<u>Bar magnets</u> - $\vec{J}=0$, $\vec{M}$ fixed and given
(<u>not</u> a linear material)

$\vec{\nabla}\cdot\vec{B}=0$

$\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\vec{J}=0$

$\vec{\nabla}\times\vec{H}=0 \Rightarrow \vec{H}=-\vec{\nabla}\Phi_M$ magnetic scalar potential

$\vec{B}=\vec{H}+4\pi\vec{M}$

$\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(\vec{H}+4\pi\vec{M})=0$

$\vec{\nabla}\cdot\vec{H}=-\nabla^2\Phi_M=-4\pi\,\vec{\nabla}\cdot\vec{M}$

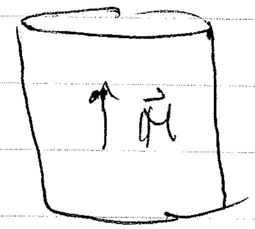
$\nabla^2\Phi_M=4\pi\,\vec{\nabla}\cdot\vec{M}$

so $\rho_M\equiv-\vec{\nabla}\cdot\vec{M}$ looks like a magnetic "charge"
ρ_M is <u>source</u> for $\vec{H}$

Also at surfaces of material $\sigma_M=\hat{n}\cdot\vec{M}$ looks like surface charge

$$\vec{H}(\vec{r})=\int_V d^3r'\,\rho_M(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}+\oint_S da'\,\sigma_M(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Field lines for $\vec{H}$ can start and end at sources and sinks given by ρ_M and σ_M



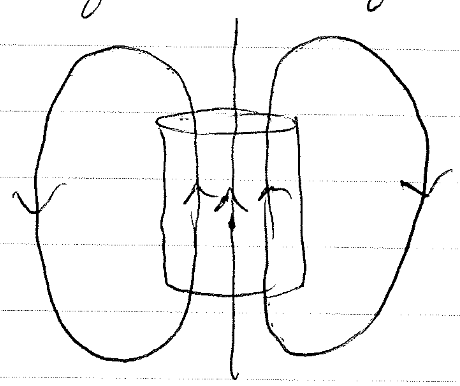
$\vec{M} = M\hat{z}$

bound currents $\vec{J}_b = c\,\vec{\nabla}\times\vec{M} = 0$

$\vec{K}_b = c\,\vec{M}\times\hat{n}$

$\vec{K}_b = \begin{cases} cM\hat{\phi} & \text{on side} \\ 0 & \text{on top + bottom} \end{cases}$

$\vec{K}_b$ is like solenoid current
field lines of $\vec{B}$ look like

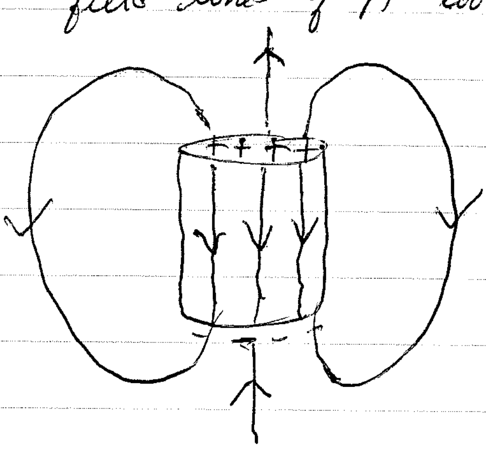


But $\vec{H}$ is determined as follows:

$\rho_M = -\vec{\nabla}\cdot\vec{M} = 0$

$\sigma_M = \hat{n}\cdot\vec{M} = \begin{cases} M & \text{on top} \\ -M & \text{on bottom} \end{cases}$

field lines of $\vec{H}$ look like parallel plate capacitor



field lines of $\vec{H}$ = field lines of $\vec{B}$ <u>outside</u> magnet, but they are very different inside the magnet!

Conservation of Energy — leave macroscopic Maxwell eqns for present. $\vec{E}$, $\vec{B}$, ρ, $\vec{j}$ are now the exact microscopic quantities

Consider a collection of charged particles, described by charge density ρ and current density $\vec{j}$. The particles are contained in a volume V.

Define $\mathcal{E}_{mech}$ as total "mechanical" energy of the particles. $\mathcal{E}_{mech}$ = sum of particles kinetic energy plus potential energy of any non electromagnetic forces.

The particles will exert forces on each other via their electromagnetic interactions, ie via the $\vec{E}$ and $\vec{B}$ fields that they create. Define W as the work done on the particles by all electromagnetic forces. Then, by the work energy theorem of mechanics:

$$\frac{d\mathcal{E}_{mech}}{dt} = \frac{dW}{dt}$$

For a single charge q_i, (at $\vec{r}_i$ with velocity $\vec{v}_i$) $\frac{dW}{dt} = \vec{F}_i \cdot \vec{v}_i$

$$= q_i \vec{E}(\vec{r}_i) \cdot \vec{v}_i + \underbrace{q\left(\frac{\vec{v}_i}{c} \times \vec{B}\right) \cdot \vec{v}_i}_{=0}$$

$$= q_i \vec{E}(\vec{r}_i) \cdot \vec{v}_i$$

For the collection of charges, with

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i \, \delta(\vec{r}-\vec{r}_i(t))$$

the total rate of work done is

$$\frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}(\vec{r}_i) = \int_V d^3r \, \vec{j} \cdot \vec{E}$$

So

$$\frac{d\mathcal{E}_{mech}}{dt} = \int_V d^3r \, \vec{j} \cdot \vec{E}$$

By maxwell equation $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$
we can write

$$\vec{j} = \frac{c}{4\pi} \left[\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right]$$

$$\int_V d^3r \, \vec{j} \cdot \vec{E} = \int_V d^3r \frac{c}{4\pi} \left[(\vec{\nabla} \times \vec{B}) \cdot E - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \cdot \vec{E} \right]$$

~~use~~ $\frac{\partial \vec{E}}{\partial t} \cdot E = \frac{1}{2} \frac{\partial E^2}{\partial t}$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot B - E \cdot (\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{B} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

then use $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

$$\text{so } \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$= -\frac{1}{2c} \frac{\partial B^2}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$